O`ZBEKISTON RESPUBLIKASI OLIY VA O`RTA MAXSUS TA'LIM VAZIRLIGI

TOSHKENT ARXITEKTURA QURULISH INSTITUTI

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Matematika 1

O`quv qo`llanma

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Annotasiya

Mazkur o‘quv qo‘llanma oliy ta’lim muassasalari barcha bakalavr ta’lim yo‘nalishlari Davlat ta’lim standartlariga va o‘quv dasturiga mos keladi. Unda Matematika fani to‘g‘risida umumiy tushunchalar va matematikaning ilmiy-texnik talqini, ishlab chiqarish jarayonining qurilish masalalarida uchraydigan ayrim metodlar keltirilgan.

O‘quv qo‘llanma barcha bakalavriat va magistratura talabalari, o‘qituvchilar hamda ilmiy tadqiqotchilar foydalanishlari uchun mo‘ljallangan.

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Аннотация

Данное учебное пособие соответствует Государственным образовательным стандартам и учебным программам всех направлений бакалавриата высших учебных заведений. В нем изложены научно-технические представления математики, некоторые методы, встречающие в производстве на примере стрительных задач.

Учебное пособие назначено студентам всех направлений бакалавриата и специальностей магистратуры, преподавательям и научным исследовательям.

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Summary

This manual complies with the State educational standards and curricula of all areas of undergraduate education. It presents scientific and technical ideas of mathematics, some methods found in production on the example of construction problems.

The manual is assigned to students of all areas of undergraduate and master’s degrees, teachers and scientific researchers.

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Kirish

Mazkur o‘quv qo‘llanma oliy ta’lim muassasalari barcha bakalavr ta’lim yo‘nalishlari Davlat ta’lim standartlariga va o‘quv dasturiga mos keladi.

Ushbu o‘quv qo‘llanmada Oliy matematika kursining chiziqli va vektorli algebra elementlari, tekisliklagi va fazodagi analitik geometriya, bir o`zgaruvchili funksiyalarning differensial hisobi bo`limlari kiritilgan. Shuningdek, o‘quv qo‘llanmada matematik tushuncha va tasdiqlarning ilmiy-texnik talqini, qurilish masalalarida uchraydigan ayrim matematik masalalar va matematik modellar keltirilgan. O‘quv qo‘llanmada oliy matematika kursini o`qitish jarayonida talabalarning mantiqiy, algoritmik, abstrakt fikrlash, matematik tafakkurni shakllantirish va rivojlantirishni, ularning fikr-mulohazalarini, xulosalarini asosli tarzda aniq bayon etishga hamda ishlab chiqarish jarayonining nazariy va amaliy masalalarini yecha olishga, yetarli matematik apparatni egallash va uni qo`llashga, qurilish masalalarining matematik modelini tuzish va tahlil qilishga o`rgatishni ko`zda tutadi.

Ushbu o‘quv qo‘llanma talabalarda matematik tafakkurni rivojlantirish, ishlab chiqarish jarayoni, jumladan, qurilishning nazariy va amaliy masalalarini yechish bo`yicha yetarli matematik bilimga ega bo`lish, ulardan foydalana olish va ularni qo`llay bilish ko`nikma va malakalarini hosil qilishdan iborat. Bunda talabalarga ko`p noma’lumli tenglamalar sistemalarini yechish, to`g`ri chiziq, tekislik va ikkinchi tartibli chiziq hamda sirtlarning tenglamalarni tuzish va ularni kanonik shaklga keltirish, differentsiallash qoidalarini hamda mazkur materiallar bo`yicha mustahkam bilim berishdan iborat.

CHIZIQLI ALGEBRA ELEMENTLARI

1-§. DETERMINANTLAR VA ULARNING XOSSALARI

Determinantlarga oid amaliy masalalar

1. Eni $5.15\,m$, balandligi $2.78\,m$ bo‘lgan devorga oboy yopishtirish kerak. Qancha gulsiz oboy talab etiladi?

   Yechish: $5.15\,m \times 2.78\,m = 14.317\,m^2$ yuza miqdorida oboy talab etiladi.

   Izoh. Bu misolni determinatlar yordamida quyidagicha yechish mumkin:

$$\begin{vmatrix} 5.15 & 0 \\ 0 & 2.78 \end{vmatrix} = 5.15 \times 2.78 = 14.317 \text{ yuza birlik (bu misolda } m^2\text{).}$$

2. Eni $5.15\,m$, balandligi $2.78\,m$ bo‘lgan devorda eshik bor. Eshikning balandligi $2.05\,m$, eni $0.80\,m$. Bu devorga qancha gulsiz oboy talab etiladi?

   Yechish: Bu devorga

$$\begin{vmatrix} 5.15 & 2.05 \\ 0.80 & 2.78 \end{vmatrix} = (5.15 \times 2.78 - 2.05 \times 0.80) = 14.317 - 1.64 = 12.677$$

yuza birligidagi miqdorida oboy talab etiladi.

3. Eni $3.20\,m$, balandligi $2.78\,m$ bo‘lgan devorda balandligi $1.65\,m$, eni $1.28\,m$ bo‘lgan deraza bor. Bu devorga qancha gulsiz oboy talab etiladi?

   Yechish: Bu devorga

$$\begin{vmatrix} 3.20 & 1.65 \\ 1.28 & 2.78 \end{vmatrix} = (3.20 \times 2.78 - 1.65 \times 1.28) = 8.896 - 2.112 = 6.784$$

yuza birligidagi (bu misolda m^2) miqdorida oboy talab etiladi.

4. Karantin davrida talaba uyida o‘lchamlari $5.15 \times 3.20\,m^2$ bo‘lgan xona devorlariga suyuq oboy yopishtirishni reja qildi. Devor balandligi $2.78\,m$ bo‘lib, uzunligi $5.15\,m$ bo‘lgan bitta devorda balandligi $2.05\,m$, eni $0.80\,m$ bo‘lgan eshik, uzunligi $3.20\,m$ bitta devorda balandligi $1.65\,m$, eni $1.28\,m$ bo‘lgan deraza bor. Qolgan ikkita devor yaxlit. Shu xonaga qancha suyuq oboy talab etiladi? Oboyning o‘rtacha qalinligini $2\,mm$ deb oling.

   Yechish: Bu devorga

$$\begin{vmatrix} 5.15 & 0 \\ 0 & 2.78 \end{vmatrix} + \begin{vmatrix} 3.20 & 1.65 \\ 1.28 & 2.78 \end{vmatrix} + \begin{vmatrix} 5.15 & 2.05 \\ 0.80 & 2.78 \end{vmatrix} + \begin{vmatrix} 3.20 & 0 \\ 0 & 2.78 \end{vmatrix}$$
$$= 14.317 + 6.784 + 12.677 + 8.896 = 42.674$$

yuza birligidagi (bu misolda m^2) miqdorida suyuq oboy talab etiladi. Uning hajmi $42.674\,m^2 \times 0.002\,m = 0.085348\,m^3 = 85.348\,dm^3$ bo‘ladi. Bir qopda $50\,dm^3$ suyuq oboy joylashsa, ikki qop suyuq oboy olish talab etiladi.

5. 2020 yilgi pandenmiya davrida masofaviy ta’lim olishga ruxsat etildi. Talaba uyida o‘lchamlari $5.15 \times 3.20\,m^2$ bo‘lgan xonaning devorlariga oboy yopishtirishni reja qildi. Devor balandligi $2.78\,m$ bo‘lib, uzunligi $5.15\,m$ bo‘lgan bitta devorda balandligi $2.05\,m$, eni $0.80\,m$ bo‘lgan eshik, uzunligi $3.20\,m$ bitta devorda balandligi $1.65\,m$, eni $1.28\,m$ bo‘lgan deraza bor. Qolgan ikkita devor yaxlit. Shu xonaga qancha oboy talab etiladi. Oboyning guli davriy bo‘lib, $60\,sm$ da takrorlanadi. Oboy gulini guliga moslab yopishtirish uchun talaba qancha oboy olishi kerak? Oboy eni $1\,m$ uzunligi $10\,m$ bo‘lgan rulonlarda sotiladi.

Yechish: Ahamiyat bering, oboy gulining davriyligi 60 sm ekan. Yonma-yon joylashgan qog‘ozlarning gullarini uyg‘unlashtirish uchun, bu qog‘ozlar 60 sm ga karrali uzunlikda qirqilishi talab etiladi. Demak, devor balandligini (oshirib) 3 m deb olishga to‘g‘ri keladi. Eshik va deraza balanliklarini esa (kamaytirib) mos ravishda 1.80 m va 1.20 m deb olishga to‘g‘ri keladi. Bu devorga

$$\begin{vmatrix} 5.15 & 0 \\ 0 & 3.00 \end{vmatrix} + \begin{vmatrix} 3.20 & 1.20 \\ 1.28 & 3.00 \end{vmatrix} + \begin{vmatrix} 5.15 & 1.80 \\ 0.80 & 3.00 \end{vmatrix} + \begin{vmatrix} 3.20 & 0 \\ 0 & 3.00 \end{vmatrix}$$
$$= 15.45 + 8.064 + 14.01 + 9.6 = 47.124$$

yuza birligidagi (bu misolda m^2) miqdorida oboy talab etiladi. Bu 5 ta oboy degani.

Bunda $5 \times 1\ m \times 10\ m - 42.674\ m^2 \approx 8\ m^2$ chiqindi chiqadi.

6. Balandligi 3.40 m, eni 12.50 m bo‘lgan devorda bitta eshik va bitta deraza bor. Eshikning eni 1.20 m, balandligi 2.10 m, derazaning o‘lchamlari esa mos ravishda 1.30 m va 1.60 m. Devorni shuvoq qilish talab etilmoqda. Ustaning ish haqi bir metr kvadratni shuvash uchun 8 000 so‘m bo‘lsa, ish yakunida ustaga qancha haq to‘lanadi?

Yechish: Devorning shuvoq qilinadigan qismining yuzasini topish uchun devor uzunligini ikki qismga, aytaylik 6 m va 6.50 m uzunlikdagi qismlarga ajratamiz. Faraz qilaylik, eshik birinchi qismda, derza esa iiknchi qismda bo‘lsin. U holda devorning

$$\begin{vmatrix} 3.40 & 1.20 \\ 2.10 & 6.00 \end{vmatrix} + \begin{vmatrix} 3.40 & 1.30 \\ 1.60 & 6.50 \end{vmatrix} = 17.88 + 20.02 = 37.9$$

yuza birlik (bu misolda m^2) dagi qismi shuvaladi. Demak, ustaning ish haqi $37.9 \times 8\ 000 = 303\ 200$ so‘m bo‘ladi.

7. Chuqurligi 3.00 m, eni 12.50 m, uzunligi 25.00 m bo‘lgan kotlovan qazishda qancha tuproq ishlari bajariladi?

Yechish: Masalani uchinchi tartibli determinant yordamida echamiz

$$\begin{vmatrix} 3.00 & 0 & 0 \\ 0 & 12.50 & 0 \\ 0 & 0 & 25.00 \end{vmatrix} = 3.00 \times 12.50 \times 25.00 = 937.50$$

hajm birligidagi (bu misolda m^3) tuproq ishlari bajariladi.

8. Eni 12.50 m, uzunligi 25.00 m bo‘lishi rejalashtirilayotgan uyning chuqurligi 3.00 m bo‘lgan tashqi fundamentini 25.00 m lik tomonlarini 60 sm qalinlikda, 12.50 m lik tomonlarni 40 sm qalinlikda qazish uchun qancha tuproq ishlari bajariladi?

Yechish: Masalani uchinchi tartibli determinant yordamida echamiz

$$\begin{vmatrix} 3.00 & 0 & 3.00 \\ 12.10 & 12.50 & 0 \\ 0 & 24.40 & 25.00 \end{vmatrix} =$$
$$= 3.00 \times 12.50 \times 25.00 - 3.00 \times 12.10 \times 24.40 = 937.50 - 885.72 = 51.78$$

hajm birligidagi (bu misolda m^3) tuproq ishlari bajariladi.

Eslatmalar. 1. SHu o‘rinda tashqi o‘lchamlari a, b, c bo‘lgan to‘g‘ri burchakli parallelepipedning a uzunlikdagi ikkita parallel yoqlarining qalinligi d_a, b yoqlarining qalinligi d_b, c yoqlarining qalinligi d_c bo‘lsa, uning barcha yoqlarning umumiy hajmi

$$V_{yoq} = \begin{vmatrix} a & 0 & a-d_a \\ b-d_b & b & 0 \\ 0 & c-d_c & c \end{vmatrix},$$

ichidagi g‘ovak hajmi esa

$$V_{g‘ovak} = \begin{vmatrix} a-d_a & 0 & 0 \\ 0 & b-d_b & 0 \\ 0 & 0 & c-d_c \end{vmatrix}$$

formula orqali topilishini eslab qoling.

Ajoyib xulosa.

$$\begin{vmatrix} a_{11}-b_1 & 0 & 0 \\ 0 & a_{22}-b_2 & 0 \\ 0 & 0 & a_{33}-b_3 \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & 0 & a_{11}-b_1 \\ a_{22}-b_2 & a_{22} & 0 \\ 0 & a_{33}-b_3 & a_{33} \end{vmatrix}.$$

2. Undan tashqari, ichida tomonlari a_1, b_1 bo‘lgan to‘g‘ri to‘rtburchak shaklidagi teshigi bor, tomonlari a, b bo‘lgan to‘g‘ri to‘rtburchak bilan chegaralangan soha yuzasi

$$S = \begin{vmatrix} a & a_1 \\ b_1 & b \end{vmatrix}$$

formula orqali topilishini eslab qoling.

Determinantlar va ularning xossalari (nazariy qism)

1.1. Ikkinchi va uchinchi tartibli determinantlar

To‘rtta a, b, c, d elementlarning tartiblangan jamlanmasiga mos qo‘yiladigan $ad - bc$ ayirma shu to‘rtta sonning determinanti deyiladi va $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ kabi belgilanadi. Demak,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc\ .$$

Bu jamlanmada tartib o‘zgarsa, aytaylik, d, a, b, c kabi bo‘lsa, bu yangi jamlanmaning determinanti

$$\begin{vmatrix} d & a \\ b & c \end{vmatrix} = cd - ab$$

bo‘ladi.

a, b elementlar birgalikda $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ determinantning birinchi satri,

c, d elementlar birgalikda $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ determinantning ikkinchi satri,

a, c elementlar birgalikda $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ determinantning birinchi ustuni,

b, d elementlar birgalikda $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ determinantning ikkinchi ustuni deb yuritiladi.

Determinantning har bir elementi nechanchi satr va nechanchi ustunda joylashganini ko‘rsatib turishi uchun unga ikkita indeks qo‘yiladi.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Bunda birinchi indeks satr nomerini, ikkinchi indeks esa ustun nomerini anglatadi. Demak,

a_{11}, a_{12} elementlar $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning birinchi satri (ikkala elementda ham birinchi indeks 1 ga teng),

a_{21}, a_{22} elementlar $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning ikkinchi satri (ikkala elementda ham birinchi indeks 2 ga teng),

a_{11}, a_{21} elementlar $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning birinchi ustuni (ikkala elementda ham ikkinchi indeks 1 ga teng),

a_{12}, a_{22} elementlar $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning ikkinchi ustuni bo‘ladi (ikkala elementda ham ikkinchi indeks 2 ga teng).

a_{11}, a_{22} elementlar birgalikda $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning bosh diagonali,

a_{12}, a_{21} elementlar birgalikda $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning yon diagonali deb yuritiladi.

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinant ikkita satr va ikkita ustundan iborat bo‘lgani uchun uni ikkinchi tartibli determinant deyiladi.

Uchta satr va uchta ustundan tashkil topgan ushbu

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

ifoda uchinchi tartibli determinant deyiladi.

Uchinchi tartibli determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11} \times a_{22} \times a_{33} + a_{12} \times a_{23} \times a_{31} + a_{13} \times a_{21} \times a_{32} -$$
$$- a_{11} \times a_{23} \times a_{32} - a_{12} \times a_{21} \times a_{33} - a_{13} \times a_{22} \times a_{31}$$

kabi hisoblanadi.

Uchinchi tartibli determinantning har bir elementiga **uning minori** deb ataluvchi, shu determinantdan shu element turgan satr va ustunni olib tashlash natijasida hosil qilingan ikkinchi tartibli determinant mos qo‘yiladi. a_{ij} elementning minori M_{ij} bo‘lsin, $i = 1, 2, 3$; $j = 1, 2, 3$. U holda, ta’rifga ko‘ra

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$
$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$
$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Uchinchi tartibli determinantning har bir a_{ij} elementiga uning M_{ij} minoridan ± 1 koeffitsientga farq qiluvchi, a_{ij} elementning **algebraik to‘ldiruvchisi** deb ataluvchi A_{ij} ifoda mos keltiriladi. Bu moslik

$$A_{ij} = (-1)^{i+j} M_{ij}$$

tenglik yordamida amalga oshiriladi. Bu erda $i = 1, 2, 3;\ j = 1, 2, 3$.

Uchinchi tartibli determinantni quyidagicha hisoblash mumkin:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13}.$$

Tenglikning o‘ng tomonidagi $a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13}$ algebraik yig‘indiga determinantning birinchi satr bo‘yicha **yoyilmasi** deyiladi. Bu yoyilmani

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

kabi yozish mumkin. Demak, uchinchi tartibli determinantni ikkinchi tartibli determinantlar yordamida hisoblash mumkin ekan.

Endi uchinchi tartibli determinantni boshqa satrlari va ustunlari bo‘yicha yoyilmasini keltiramiz.

Ikkinchi satr bo‘yicha yoyilmasi

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$

Uchinchi satr bo‘yicha yoyilmasi

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

Birinchi ustun bo‘yicha yoyilmasi

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix},$$

Ikkinchi ustun bo‘yicha yoyilmasi

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{22} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix},$$

Uchinchi ustun bo‘yicha yoyilmasi

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

1.2. Yuqori tartibli determinantlar

Ushbu

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

ifoda n inchi tartibli determinant deyiladi. n-tartibli determinantni hisoblash uchun u ixtiyoriy bitta satri (yoki ixtiyoriy bitta ustuni) bo‘yicha yoyiladi. YOyilmadagi har bir $(n-1)$-tartibli determinant yana ixtiyoriy bitta satri (yoki ixtiyoriy bitta ustuni) bo‘yicha yoyiladi. Bu jarayon yoyilmada uchinchi tartibli determinant hosil bo‘lguncha davom ettiriladi va natijada, n-tartibli determinantning qiymati topiladi.

Masalan, to‘rtinchi tartibli determinantni, aytaylik, uchinchi ustun bo‘yicha quyidagicha yoyish mumkin

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

$$= a_{13} \cdot (-1)^{1+3} \cdot \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + a_{23} \cdot (-1)^{2+3} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + a_{33} \cdot (-1)^{3+3} \cdot \begin{vmatrix} a_{11} & a_{22} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + a_{34} \cdot (-1)^{4+3} \cdot \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} =$$

$$= a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{22} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{34} \cdot \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}.$$

Shunday qilib, to‘rtinchi tartibli determinantni hisoblash har bitta satri (yoki har bitta ustuni) bo‘yicha yoyilganda to‘rtta uchinchi tartibli determinantni hisoblashga keltirilar ekan. Bunda satr yoki ustun bo‘yicha jami yoyilmalar soni $2 \cdot 3 = 6$ ta bo‘ladi.

Savol: 1) Beshinchi tartibli determinantni uning bitta yoyilmasida nechta uchinchi tartibli determinantlar ishtirok etadi? 2) Beshinchi tartibli determinant nechta usulda uchinchi tartibli determinantlarga yoyiladi?

Javob: 1) Avvalo beshinchi tartibli determinant biror tayinli satri (yoki ustuni) bo‘yicha **beshta** to‘rtinchi tartibli determinantlarga yoyilishii ta’kidlaymiz. Hosil bo‘lgan bu to‘rtinchi tartibli determinantlarning har biri tayinli satr (yoki ustun) lar bo‘yicha **to‘rtta**dan uchinchi tartibli determinantlarga yoyiladi. Demak, beshinchi tartibli determinantning bitta yoyilmasida $5 \cdot 4 = 20$ ta uchinchi tartibli determinantlar ishtirok etishi mumkin ekan. Bu degani beshinchi tartibli determinant o‘zining tayinli yoyilmasida (yoyilmadagi ayrim elementlar nolga teng bo‘lishini e’tiborga olsak) eng ko‘pi bilan 20 ta uchinchi tartibli determinantlarning yig‘indisi shaklida ifodalanadi.

2) Beshinchi tartibli determinant (5 ta satri va 5 ta ustuni bo‘lgani uchun) $2 \cdot 5 = 10$ ta usulda to‘rtinchi tartibli determinantlarga yoyiladi. Har bitta yoyilmada 5 tadan to‘rtinchi tartibli determinantlar ishtirok etishi mumkin. Bu to‘rtinchi tartibli determinantlarning har biri $2 \cdot 4 = 8$ ta usulda uchinchi tartibli determinantlarga yoyiladi. 5 ta to‘rtinchi tartibli determinantlar esa 8^5 usulda uchinchi tartibli determinantlarga yoyiladi.

Demak, beshinchi tartibli determinant $10 \cdot 8^5$ ta usulda uchinchi tartibli determinantlarga yoyilishi mumkin ekan. Endi bu yoyilmalarning eng maqbulini tanlash masalasi paydo bo‘ladi.

Oltinchi tartibli determinantning bitta yoyilmasida $6 \cdot 5 \cdot 4 = 120$ ta uchinchi tartibli determinantlar ishtirok etadi (**nega?**).

Umuman olganda, n-tartibli determinantning bitta yoyilmasida

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 5 \cdot 4 = \frac{n!}{3!}$$

ta uchinchi tartibli determinantlar ishtirok etishi mumkin ($n \geq 3$) (**isbotlang!**). Bu erda $n!$ ifoda "en-faktorial" deb o'qiladi va quyidagicha topiladi

$$n! = 1 \cdot 2 \cdot \ldots \cdot n.$$

$0! = 1$ deb qabul qilamiz (Kombinatorikada $0! = 1$ tenglikning to'g'riligi isbotlanadi, bu isbot bizdan talab etilmaydi. Hohlovchilar uni isbotlashi mumkin).

$$1! = 1, \quad 2! = 1 \cdot 2 = 2, \quad 3! = 1 \cdot 2 \cdot 3 = 6,$$
$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24, \quad 5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

bo'lishini ko'rish mumkin.

Savol. n-tartibli determinantning bitta yoyilmasida nechta ikkinchi tartibli determinantlar ishtirok etishi mumkin?

Xulosa qilib shuni ta'kidlash mumkinki, yuqori tartibli determinantlarni yoyilmalar orqali hisoblash ko'plab hisob-kitoblarni talab qilar ekan. Bunday murakkab vaziyatdan determinantlarning xossalarini o'rganish orqali chiqish mumkin.

1.3. Determinantlarning xossalari

1. Determinantning tayinli biror satri yoki ustunining barcha elementlarini biror songa ko'paytirish mazkur determinantni shu songa ko'paytirishni anglatadi. Masalan:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda a_{i1} & \lambda a_{i2} & \cdots & \lambda a_{ij} & \cdots & \lambda a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} = \lambda \cdot \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix},$$

yoki

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & \lambda a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \lambda a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & \lambda a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \lambda a_{nj} & \cdots & a_{nn} \end{vmatrix} = \lambda \cdot \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}.$$

Determinantning bu xossasi, xususan, umumiy ko'paytuvchini determinant oldiga chiqarish imkonini beradi.

2. Determinantning ixtiyoriy ikkita satrlari (yoki ixtiyoriy ikkita ustunlari)ning o'rinlari almashtirilsa, determinantning ishorasi qarama-qarshisiga o'zgaradi:

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = - \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

yoki

$$\begin{vmatrix} \cdots & a_{1l} & \cdots & a_{1j} & \cdots \\ \cdots & a_{2l} & \cdots & a_{2j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{nl} & \cdots & a_{nj} & \cdots \end{vmatrix} = - \begin{vmatrix} \cdots & a_{1j} & \cdots & a_{1l} & \cdots \\ \cdots & a_{2j} & \cdots & a_{2l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{nj} & \cdots & a_{nl} & \cdots \end{vmatrix}.$$

3. Determinant biror satri (yoki biror ustuni) faqatgina nollardan iborat bo‘lsa, bu determinant nolga teng:

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)n} \\ 0 & 0 & \cdots & 0 \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)n} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \ \ (i\text{-satr nollardan iborat}),$$

yoki

$$\begin{vmatrix} \cdots & a_{1(j-1)} & 0 & a_{1(j+1)} & \cdots \\ \cdots & a_{2(j-1)} & 0 & a_{2(j+1)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{n(j-1)} & 0 & a_{n(j+1)} & \cdots \end{vmatrix} = 0 \ (j\text{-ustun nollardan iborat}).$$

Faqat nollardan iborat satr nol-satr, faqat nollardan iborat ustun nol-ustun deb yuritiladi.

4. Ikkita satri (yoki ikkita ustuni) teng bo‘lsa, determinant nolga teng bo‘ladi:

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \ .$$

yoki

$$\begin{vmatrix} \cdots & a_{1l} & \cdots & a_{1l} & \cdots \\ \cdots & a_{2l} & \cdots & a_{2l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{nl} & \cdots & a_{nl} & \cdots \end{vmatrix} = 0 \ .$$

5. Ikkita satri (yoki ikkita ustuni) proporsional bo‘lsa, determinant nolga teng bo‘ladi:

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{k1} & \lambda a_{k2} & \cdots & \lambda a_{kn} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \ .$$

yoki

$$\begin{vmatrix} \cdots & a_{1l} & \cdots & \lambda a_{1l} & \cdots \\ \cdots & a_{2l} & \cdots & \lambda a_{2l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{nl} & \cdots & \lambda a_{nl} & \cdots \end{vmatrix} = 0 \ .$$

6. Ushbu ko‘rinishdagi

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} \quad \text{yoki} \quad \begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ a_{n1} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{vmatrix}$$

determinant yuqori (yoki quyi) uchburchak-determinant deyiladi va u o‘zining bosh diagonalidagi elementlari ko‘paytmasiga teng:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}; \quad \begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ a_{n1} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn} \ .$$

7. Determinant biror i-satrining har bir elementi $a_{ik} = b_{ik} \pm c_{ik}$ ko‘rinishda (yoki biror j-ustunining har bir elementi $a_{lj} = b_{lj} \pm c_{lj}$ ko‘rinishda) bo‘lsa, u holda quyidagi tengliklar o‘rinli

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{i1} \pm c_{i1} & b_{i2} \pm c_{i2} & \cdots & b_{ij} \pm c_{ij} & \cdots & b_{in} \pm c_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{ij} & \cdots & b_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} \pm \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix},$$

yoki

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & b_{ij} \pm c_{ij} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_{ij} \pm c_{ij} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & b_{ij} \pm c_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & b_{ij} \pm c_{ij} & \cdots & a_{nn} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{ij} & \cdots & b_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} \pm \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}.$$

8. Determinantning biror i-satrining har bir elementini biror λ songa ko‘paytirib, boshqa biror k-satriga hadma-had qo‘shish natijasida hosil bo‘lgan determinant dastlabki determinantga teng:

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{i1} + a_{k1} & \lambda a_{i2} + a_{k1} & \cdots & \lambda a_{in} + a_{k1} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

Yoki determinantning biror j-satrining har bir elementini biror λ songa ko‘paytirib, boshqa biror l-satriga hadma-had qo‘shish natijasida hosil bo‘lgan determinant dastlabki determinantga teng:

$$\begin{vmatrix} \cdots & \lambda a_{1j} + a_{1l} & \cdots & a_{1j} & \cdots \\ \cdots & \lambda a_{2j} + a_{2l} & \cdots & a_{2j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \lambda a_{nj} + a_{nl} & \cdots & a_{nj} & \cdots \end{vmatrix} = \begin{vmatrix} \cdots & a_{1l} & \cdots & a_{1j} & \cdots \\ \cdots & a_{2l} & \cdots & a_{2j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{nl} & \cdots & a_{nj} & \cdots \end{vmatrix}.$$

Mustaqil ishlash uchun topshiriq. Determinantlarning xossalarining uchinchi va to‘rtinchi tartibli determinantlar uchun bajarilishini tekshiring.

Mustahkamlash uchun mashqlar

Determinantlarga oid masalalar

1. Ushbu tengliklarni isbotlang.

1) $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = 1,$

2) $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos 2\alpha,$

3) $\begin{vmatrix} \sin\alpha & -\sin\alpha \\ \cos\alpha & \cos\alpha \end{vmatrix} = \sin 2\alpha,$

4) $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix} = \cos(\alpha+\beta),$

5) $\begin{vmatrix} \sin\alpha & \cos\alpha \\ \cos\beta & \sin\beta \end{vmatrix} = \sin(\alpha-\beta),$

6) $\begin{vmatrix} \sin\alpha & -\cos\alpha \\ \cos\beta & \sin\beta \end{vmatrix} = \sin(\alpha+\beta),$

7) $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix} = \cos(\alpha-\beta),$

8) $\begin{vmatrix} \operatorname{tg}\alpha & 0 \\ 0 & \operatorname{ctg}\alpha \end{vmatrix} = 1,$

9) $\begin{vmatrix} \operatorname{tg}\alpha & -1 \\ 1 & \operatorname{tg}\alpha \end{vmatrix} = \frac{1}{\cos^2\alpha},$

10) $\begin{vmatrix} \cos^2\alpha & \sin^2\alpha \\ \sin^2\alpha & \cos^2\alpha \end{vmatrix} = \cos 2\alpha,$

11) $\begin{vmatrix} a & b \\ b & a \end{vmatrix} = (a-b)(a+b),$

12) $\begin{vmatrix} a & -b \\ b & a \end{vmatrix} = a^2+b^2,$

13) $\begin{vmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{vmatrix} = (a+b)(a^2-ab+b^2),$

14) $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = (a-b)(a^2+ab+b^2),$

15) $\begin{vmatrix} 0 & a & b \\ b & 0 & a \\ a & b & 0 \end{vmatrix} = (a+b)(a^2-ab+b^2),$

16) $\begin{vmatrix} a & b & 0 \\ b & 0 & a \\ 0 & a & b \end{vmatrix} = -a^3-b^3,$

2. Determinantlarni hisoblang.

1) $\begin{vmatrix} 3 & 1 & -1 \\ 2 & 5 & 0 \\ 0 & -3 & 2 \end{vmatrix},$

2) $\begin{vmatrix} -1 & 1 & 3 \\ 0 & 5 & 2 \\ 2 & -3 & 0 \end{vmatrix},$

3) $\begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & -1 \\ 0 & -3 & 2 \end{vmatrix},$

4) $\begin{vmatrix} 2 & 1 & 3 \\ 0 & 5 & 2 \\ -4 & -3 & 0 \end{vmatrix},$

5) $\begin{vmatrix} 2 & 1 & -1 \\ -3 & 5 & 0 \\ 3 & -3 & 2 \end{vmatrix},$

6) $\begin{vmatrix} 3 & -5 & 3 \\ 0 & 5 & 2 \\ 2 & -3 & 0 \end{vmatrix},$

7) $\begin{vmatrix} 2 & 1 & -1 \\ 2 & 5 & 0 \\ 2 & -3 & 2 \end{vmatrix},$

8) $\begin{vmatrix} -1 & 4 & 3 \\ 0 & 7 & 2 \\ 2 & -3 & 0 \end{vmatrix},$

9) $\begin{vmatrix} 2 & 5 & 5 \\ 3 & 1 & 0 \\ 0 & -3 & -1 \end{vmatrix},$

10) $\begin{vmatrix} 7 & 1 & 3 \\ 0 & 5 & 2 \\ -19 & -3 & 0 \end{vmatrix},$

11) $\begin{vmatrix} 2 & -4 & -1 \\ -3 & 5 & 0 \\ 3 & 7 & 2 \end{vmatrix},$

12) $\begin{vmatrix} 18 & -5 & 3 \\ 10 & 5 & 2 \\ 2 & -3 & 0 \end{vmatrix},$

13) $\begin{vmatrix} 3 & 1 & -1 \\ 2 & -1 & 4 \\ 0 & -3 & 2 \end{vmatrix},$

14) $\begin{vmatrix} -1 & 1 & 3 \\ 6 & -4 & 2 \\ 2 & -3 & 0 \end{vmatrix},$

15) $\begin{vmatrix} 2 & -11 & 8 \\ 3 & 1 & -1 \\ 0 & -3 & 2 \end{vmatrix},$

16) $\begin{vmatrix} 2 & -4 & -1 \\ 0 & 5 & 2 \\ -4 & -3 & 0 \end{vmatrix},$

3. Determinantning ko‘rsatilgan elementining minorini hisoblang.

$$A = \begin{vmatrix} 2 & 5 & 1 & 3 \\ -1 & 11 & 3 & 7 \\ 3 & 0 & -5 & 1 \\ 11 & 7 & 5 & 1 \end{vmatrix}.$$

1) a_{11}, 2) a_{12}, 3) a_{13}, 4) a_{14}, 5) a_{21}, 6) a_{22}, 7) a_{23}, 8) a_{24},

9) a_{31}, 10) a_{32}, 11) a_{33}, 12) a_{34}, 13) a_{41}, 14) a_{42}, 15) a_{43}, 16) a_{44} .

4. Determinantning ko‘rsatilgan elementining algebraik to‘ldirmasini hisoblang.

$$A = \begin{vmatrix} 2 & 1 & 3 & -4 \\ 1 & -3 & 0 & 2 \\ 4 & 2 & -1 & 3 \\ -2 & 4 & 3 & 1 \end{vmatrix}.$$

1) a_{11}, 2) a_{12}, 3) a_{13}, 4) a_{14}, 5) a_{21}, 6) a_{22}, 7) a_{23}, 8) a_{24},

9) a_{31}, 10) a_{32}, 11) a_{33}, 12) a_{34}, 13) a_{41}, 14) a_{42}, 15) a_{43}, 16) a_{44} .

5. Determinantning ko‘rsatilgan satr yoki ustun bo‘yicha yoyilmasini yozing.

$$A = \begin{vmatrix} 3 & 1 & 2 & -4 \\ -1 & 2 & 4 & 3 \\ 0 & -3 & 1 & 2 \\ 3 & 4 & -2 & 1 \end{vmatrix}.$$

1) Birinchi satr bo‘yicha, 2) Ikkinchi satr bo‘yicha, 3) Uchinchi satr bo‘yicha,
4) To‘rtinchi satr bo‘yicha, 5) Birinchi ustun bo‘yicha, 6) Ikkinchi ustun bo‘yicha,
7) Uchinchi ustun bo‘yicha, 8) To‘rtinchi ustun bo‘yicha.

O`z-o`zini tekshirish uchun savollar

1. Ikkinchi tartibli determinant deb nimaga aytiladi?
2. Uchinchi tartibli determinant deb nimaga aytiladi?
3. Ikkinchi va uchinchi tartibli determinantlarning qanday xossalarini bilasiz?
4. Yuqori tartibli determinantlarni qanday hisoblash mumkin?
5. Minor deb nimaga aytiladi?
6. Algebraik to`ldiruvchi deb nimaga aytiladi?

O`z-o`zini tekshirish uchun testlar

Determinantlarga oid testlar

1. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = determinantning tartibi nechaga teng?

a) 3 b) 2 c) 4 d) 1

2. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning birinchi satri elementlarni ko‘rsating.

a) a_{11}, a_{12} b) a_{21}, a_{22} c) a_{11}, a_{21} d) a_{12}, a_{21}

3. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning ikkinchi satri elementlarni ko‘rsating.

a) a_{11}, a_{12} b) a_{21}, a_{22} c) a_{11}, a_{21} d) a_{12}, a_{21}

4. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning birinchi ustuni elementlarni ko‘rsating.

a) a_{11}, a_{12} b) a_{21}, a_{22} c) a_{11}, a_{21} d) a_{12}, a_{21}

5. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning ikkinchi ustuni elementlarni ko‘rsating.

a) a_{11}, a_{12} b) a_{21}, a_{22} c) a_{11}, a_{21} d) a_{12}, a_{22}

6. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning diagonali elementlarni ko‘rsating.

a) a_{11}, a_{12} b) a_{11}, a_{22} c) a_{11}, a_{21} d) a_{12}, a_{21}

7. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ determinantning diagonali elementlarni ko‘rsating.

a) a_{11}, a_{12} b) a_{11}, a_{22} c) a_{11}, a_{21} d) a_{12}, a_{21}

8. $\begin{vmatrix} 1 & 6 \\ -2 & 4 \end{vmatrix}$ determinantni hisoblang.

a) 16 b) 12 c) -8 d) 12

9. $\begin{vmatrix} a & b \\ b & a \end{vmatrix} =$ determinantni hisoblang.

a) $(a-b)(a+b)$ b) $2ab$ c) $-2ab(a-b)$ d) $(ab)^2$

10. $\begin{vmatrix} -2 & -3 \\ 5 & 4 \end{vmatrix}$ determinantni hisoblang.

a) 6 b) 7 c) -8 d) 12

11. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ determinantning tartibi nechaga teng?

a) 3 b) 2 c) 4 d) 1

12. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$ determinant qaysi formula bilan topiladi?

a) $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

b) $-a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}$

c) $a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

d) $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{21}a_{32}$

13. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$ determinantning a_{21} elementining minori toping.

a) $\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$ b) $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ c) $\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ d) $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

14. Uchinchi tartibli determinantning a_{ij} elementning minori qanday belgilanadi?

a) M_{ij} b) A_{ij} c) A_{ji} d) M_{ji}

15. Uchinchi tartibli determinantning a_{ij} elementning minori deb …

a) shu determinantdan shu element turgan satr va ustunni olib tashlash natijasida hosil qilingan ikkinchi tartibli determinantga aytiladi.
b) shu determinantda shu element turgan satrg va ustun elementlarning o‘rniga 1 qo‘yishdan natijasida hosil qilingan uchinchi tartibli determinantga aytiladi.
c) shu determinantda shu element turgan satrg va ustun elementlarning o‘rniga -1 qo‘yishdan natijasida hosil qilingan uchinchi tartibli determinantga aytiladi.
d) shu determinantdan shu element turgan satr va ustunni olib tashlash natijasida hosil qilingan ikkinchi tartibli determinantni -1 ga ko‘paytmasiga aytiladi.

16. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$ determinantning a_{ij} elementining algebraik to‘ldiruvchisini toping.

a) $A_{ij} = (-1)^{i+j} M_{ij}$ b) $A_{ij} = (-1)^{ij} M_{ij}$ c) $A_{ij} = (-1)^{i} M_{ij}$ d) $A_{ij} = (-1)^{j} M_{ij}$

17. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$ uchinchi tartibli determinantni birinchi satr bo‘yicha yoyilmasini ko‘rsating.

a) $= a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13}$
b) $= a_{11} \cdot A_{11} - a_{12} \cdot A_{12} + a_{13} \cdot A_{13}$
c) $= a_{11} \cdot A_{11} + a_{12} \cdot A_{12} - a_{13} \cdot A_{13}$
d) $= a_{11} \cdot A_{11} - a_{12} \cdot A_{12} - a_{13} \cdot A_{13}$

18. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$ uchinchi tartibli determinantni ikkinchi satri bo‘yicha yoyilmasini ko‘rsating.

a) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

b) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{22} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

c) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

d) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

19. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$ uchinchi tartibli determinantni ikkinchi ustuni bo‘yicha yoyilmasini ko‘rsating.

a) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

b) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{22} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

c) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

d) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

20. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$ uchinchi tartibli determinantni uchinchi satri bo‘yicha yoyilmasini ko‘rsating.

a) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

b) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{22} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

c) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

d) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

21. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$ uchinchi tartibli determinantni uchinchi ustuni bo‘yicha yoyilmasini ko‘rsating.

a) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

b) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{22} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

c) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

d) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

22. Agar determinantning barcha satrlari mos ustunlari bilan almashtirilsa...

a) uning qiymati o‘zgarmaydi

b) uning qiymati qarama-qarshi ishoraga o‘zgardi

c) uning qiymati nolga teng

d) uning qiymati birga teng

23. Determinantning tayinli biror satri yoki ustunining umumiy ko‘paytuvchisini…

a) determinant belgisidan tashqariga chiqarish mumkin
b) determinant belgisidan tashqariga chiqarish mumkin emas
c) determinantning diagonali elementlariga bo‘lish mumkin
d) determinantning diagonali elementlariga ko‘paytirish mumkin

24. Determinantning ixtiyoriy ikkita satrlari (yoki ixtiyoriy ikkita ustunlari)ning o‘rinlari almashtirilsa...
a) determinantning ishorasi qarama-qarshisiga o‘zgaradi
b) determinantning qiymati o‘zgarmaydi
c) determinant nolga teng bo‘ladi
d) determinant birga teng bo‘ladi

25. Determinant biror satri (yoki biror ustuni) faqatgina nollardan iborat bo‘lsa…
a) determinantning ishorasi qarama-qarshisiga o‘zgaradi
b) determinantning qiymati o‘zgarmaydi
c) determinant nolga teng bo‘ladi
d) determinant birga teng bo‘ladi

26. Ikkita satri (yoki ikkita ustuni) teng bo‘lsa…
a) determinantning ishorasi qarama-qarshisiga o‘zgaradi
b) determinantning qiymati o‘zgarmaydi
c) determinant nolga teng bo‘ladi
d) determinant birga teng bo‘ladi

26. Ikkita satri (yoki ikkita ustuni) proporsional bo‘lsa...
a) determinantning ishorasi qarama-qarshisiga o‘zgaradi
b) determinantning qiymati o‘zgarmaydi
c) determinant nolga teng bo‘ladi
d) determinant birga teng bo‘ladi

27. $\begin{vmatrix} 1 & -2 & 3 \\ -2 & 1 & -5 \\ 3 & 2 & 7 \end{vmatrix}$ = uchinchi tartibli determinantni hisoblang.
a) 16 b) -2 c) -8 d) 12

28. $\begin{vmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & 2 & 7 \end{vmatrix}$ = uchinchi tartibli determinantni hisoblang.
a) 16 b) -2 c) 0 d) 12

29. $\begin{vmatrix} 1 & -2 & 3 \\ 5 & -10 & -6 \\ 0 & 0 & 7 \end{vmatrix}$ = uchinchi tartibli determinantni hisoblang.
a) 6 b) -2 c) 0 d) 12

30. $\begin{vmatrix} 1 & -2 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & 7 \end{vmatrix}$ = uchinchi tartibli determinantni hisoblang.
a) 6 b) -7 c) 0 d) 12

2-§. MATRITSALAR VA ULAR USTIDA AMALLAR. MATRITSA DETERMINANTI. MATRITSA RANGI

Matritsalarga oid amaliy masalalar

1. Zichligi $7\,800\frac{kg}{m^3}$ bo‘lgan sementdan $5\,m^3$, zichligi $4\,200\,\frac{kg}{m^3}$ bo‘lgan qumdan $15\,m^3$, zichligi $5\,400\,\frac{kg}{m^3}$ bo‘lgan shag‘aldan $15\,m^3$ aralashtirib, beton tayyorlandi. Beton mutlaq quritilgach, uning massasi qancha bo‘ladi?

   Yechish. Zichlikning satr-matritsasi $(7800,\ 4200,\ 5400)\,\frac{kg}{m^3}$, hajmning ustun-matritsasi $\begin{pmatrix}5\\15\\15\end{pmatrix}\,m^3$ bo‘ladi. Beton massasi

$$m = (7800, 4200, 5400)\cdot\begin{pmatrix}5\\15\\15\end{pmatrix} = 7800\times 5 + 4200\times 15 + 5400\times 15 =$$
$$= 183\,000\,kg = 183\,t$$

   ekan.

2. Toshkent siti qurilishida ishlab chiqarish amaliyotida bo‘lgan 1-talaba amaliyot davrida zichligi $7\,800\frac{kg}{m^3}$ bo‘lgan sementdan $5\,m^3$, zichligi $4\,200\,\frac{kg}{m^3}$ bo‘lgan qumdan $10\,m^3$, zichligi $5\,400\,\frac{kg}{m^3}$ bo‘lgan shag‘aldan $15\,m^3$ aralashtirib, biror markali beton tayyorladi. Boshqa talaba esa zichligi $7\,500\frac{kg}{m^3}$ bo‘lgan sementdan $4\,m^3$, zichligi $4\,800\,\frac{kg}{m^3}$ bo‘lgan qumdan $12\,m^3$, zichligi $5\,600\,\frac{kg}{m^3}$ bo‘lgan shag‘aldan $12\,m^3$ aralashtirib, boshqa markali beton tayyorladi. Bu betonlar mutlaq quritilgach, ularning har birining massasi qancha bo‘ladi?

   Yechish. Birinchi va ikkinchi markali betonlarning zichliklarining matritsasi $\begin{pmatrix}7800 & 4200 & 5400\\7500 & 4800 & 5600\end{pmatrix}$, hajmlarining matritsasi esa $\begin{pmatrix}5 & 4\\10 & 12\\15 & 12\end{pmatrix}$ bo‘ladi. Betonlar massalari

$$\begin{pmatrix}7800 & 4200 & 5400\\7500 & 4800 & 5600\end{pmatrix}\cdot\begin{pmatrix}5 & 4\\10 & 12\\15 & 12\end{pmatrix} = \begin{pmatrix}7800\cdot 5 + 4200\cdot 10 + 5400\cdot 15\\7500\cdot 4 + 4800\cdot 12 + 5600\cdot 12\end{pmatrix} =$$
$$= \begin{pmatrix}162000\\154800\end{pmatrix}kg = \begin{pmatrix}162.0\\154.8\end{pmatrix}\,t.$$

   Javob. 1-markali beton 162 t., 2-markali 154.8 t.

3. Hovli qurmoqchi bo‘lgan buyurtmachidan pudratchi kelishilgan mablag‘ni oldindan oldi. Hovli yerto‘lasi uchun eni $12\,m$, uzunligi $20\,m$, chuqurligi $3.20\,m$ bo‘lgan tuproq qazish ishlari bor. Qazib chiqariladigan tuproqning o‘rtacha zichligi $2\,200\frac{kg}{m^3}$. MAN mashinasi $25\,tonna$, KamAZ esa $8\,tonna$ yuk tashishga mo‘ljallangan bo‘lib, MAN mashinasining bitta reys uchun xizmat haqi $325\,000\,so‘m$, KamAZ mashinasiniki esa $156\,000\,so‘m$.

   Siz pudratchi bo‘lsangiz qaysi transportni tanlagan bo‘lar edingiz?

Yechish. Avvalo tuproq ishlari hajmi topiladi: $V = 12\,m \times 20\,m \times 3.20\,m = 768\,m^3$. Chiqariladigan tuproq massasi $m = 2\,200 \times 768\,kg = 1\,689\,600\,kg = 1\,689.6\,t$. Demak, Man avtomobili $\frac{1\,689.6}{25} = 67.564 \approx 68$ ta reys, KamAZ avtomobili $\frac{1\,689.6}{8} = 211.2 \approx 211$ ta reys amalga oshirishi talab etiladi. Demak, reyslar matritsasi $reys\begin{pmatrix}\text{Man}\\ \text{KamAZ}\end{pmatrix} = \begin{pmatrix}68 & 0\\ 0 & 211\end{pmatrix}$ ga ega bo‘ldik.

Bitta reysning narxlari $narx\begin{pmatrix}\text{Man}\\ Kamaz\end{pmatrix} = \begin{pmatrix}325\,000\\ 156\,000\end{pmatrix}$. Ish yakuni bo‘yicha quyidagiga ega bo‘lamiz:

$$xizmat\ haqi\begin{pmatrix}\text{Man}\\ \text{KamAZ}\end{pmatrix} = \begin{pmatrix}68 & 0\\ 0 & 211\end{pmatrix} \times \begin{pmatrix}325\,000\\ 156\,000\end{pmatrix} = \begin{pmatrix}22\,100\,000\\ 32\,916\,000\end{pmatrix},$$

ya’ni ish yakuni bo‘yicha Man avtomobili xizmat haqi 22 100 000 so‘m, KamAZ avtomobili xizmat haqi 32 916 000 so‘m bo‘lar ekan. Man avtomobili ishlatilganda qariyb 11 000 000 so‘m tejalar ekan.

4. Standart g‘isht o‘lchamlari $25\,sm \times 12.5\,sm \times 6.25\,sm$ bo‘ladi. Uzunligi $22\,m$, balandligi $3.20\,m$, qalinligi $0.40\,m$ bo‘lgan yaxlit devor qurish uchun usta uy egasidan uzunasiga $\frac{2200}{25} = 88$ ta, bo‘yiga $\frac{320}{6.25} = 51.2 \approx 51$ ta, eniga $\frac{40}{12.5} = 3.2 \approx 3$, jami $88 \times 51 \times 3 = 13\,464$ ta g‘isht uchun pul oldi. Ish yakunlangach, sarflangan g‘isht soni boshqacha chiqdi. Vaziyatni qanday tushuntirasiz?

Yechish. Masalani matritsa usulida ishlaymiz. G‘ishtlar orasiga qalinligi $x\,sm$ loy qo‘yilgan bo‘lsin. U holda sarflanadigan g‘ishtlar matritsasi

$$\begin{pmatrix}\frac{2200}{25+x} & 0 & 0\\ 0 & \frac{320}{6.25+x} & 0\\ 0 & 0 & \frac{40}{12.5+x}\end{pmatrix}$$

ko‘rinishda bo‘ladi. Uning determinanti sarflanadigan g‘isht sonini aniqlaydi:

$$det\begin{pmatrix}\frac{2200}{25+x} & 0 & 0\\ 0 & \frac{320}{6.25+x} & 0\\ 0 & 0 & \frac{40}{12.5+x}\end{pmatrix} = \frac{2200}{25+x} \times \frac{320}{6.25+x} \times \frac{40}{12.5+x}.$$

Aytalik loy qalinligi 1 sm bo‘lsin. U holda aslida

$$det\begin{pmatrix}\frac{2200}{25+1} & 0 & 0\\ 0 & \frac{320}{6.25+1} & 0\\ 0 & 0 & \frac{40}{12.5+1}\end{pmatrix} = \frac{2200}{26} \times \frac{320}{7.25} \times \frac{40}{13.5} \approx 84{,}5 \times 44 \times 3 = 11\,154 \text{ ta}$$

g‘isht teriladi. Bu holda $13\,464 - 11\,154 = 2\,310$ ta g‘ishtga ortiqcha pul sarflashgan. Pishiq g‘ishtning o‘rtacha narxi $1\,000\ so‘m$ bo‘lsa, u holda uy egasi $2\,310 \times 1\,000 = 2\,310\,000$ so‘m ortiqcha xarajat qilgan.

Matritsalar va ular ustida amallar.
Matritsa determinanti. Matritsa rangi (nazariy qism)

2.1. Matritsa tushunchasi

m va n lar natural sonlar bo‘lsin. Ixtiyoriy tabiatli elementlardan tashkil topgan ushbu

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

jadval **matritsa** deb yuritiladi. Matritsalar lotin alifbosining bosh bosma A, B, $C\dots$ harflari orqali belgilanadi. Demak,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

A matritsadagi:

a_{11} a_{12} ... a_{1n} elementlar jamlanmasi A matritsaning 1-satri (bu elementlarning barchasida birinchi indeks 1 ga teng),

a_{21} a_{22} ... a_{2n} elementlar jamlanmasi A matritsaning 2-satri (bu elementlarning barchasida birinchi indeks 2 ga teng),

..., ...,

a_{m1} a_{m2} ... a_{mn} elementlar jamlanmasi A matritsaning m-satri (bu elementlarning barchasida birinchi indeks m ga teng),

a_{11}, a_{21}, $\vdots$, a_{m1} elementlar jamlanmasi A matritsaning 1-ustuni (bu elementlarning barchasida ikkinchi indeks 1 ga teng),

a_{12}, a_{22}, $\vdots$, a_{m2} elementlar jamlanmasi A matritsaning 2-ustuni (bu elementlarning barchasida ikkinchi indeks 2 ga teng),

..., ...,

a_{1n}, a_{2n}, $\vdots$, a_{mn} elementlar jamlanmasi A matritsaning n-ustuni (bu elementlarning barchasida ikkinchi indeks n ga teng)

deb yuritiladi.

Demak, qaralayotgan A matritsa m ta satr va n ustunga ega ekan. m ta satr va n ustunga ega matritsaning o‘lchami $m \times n$ ga teng deb yuritiladi va $[A] = m \times n$ kabi belgilanadi. A matritsaning a_{ij} elementining indeksidagi birinchi son (i soni)

satr nomerini, ikkinchi son (j soni) ustun nomerini anglatadi. a_{ij} element A matritsaning i-satri bilan j-ustuni kesishmasida joylashadi.

Agar matritsada satrlar soni ustunlar soni teng bo‘lsa, u holda bunday matritsalar **kvadrat matritsalar** deb yuritiladi. Kvadrat matritsalarning umumiy ko‘rinishi

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}$$

kabi bo‘ladi, $m = 1, 2, 3, \dots$.

Kvadrat matritsalarda diagonal tushunchasini kiritish mumkin:
a_{11}, a_{22}, $\dots$, a_{mm} elementlar jamlanmasi A matritsaning *bosh diagonali*,
a_{1m}, $a_{2(m-1)}$, $\dots$, a_{m1} elementlar jamlanmasi A matritsaning *yon diagonali* deyiladi.

Kvadrat matritsaning bosh diagonali faqat 1 lardan tashkil topgan bo‘lib, matritsaning qolgan elementlari 0 lardan iborat bo‘lsa, bu kvadrat matritsa **birlik matritsa** deyiladi. Birlik matritsa

$$E = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

kabi belgilanadi.

2.2. Matritsalar ustida amallar

Matritsalar ustida bajariladigan quyidagi amallarni o‘rganamiz: matritsalarni qo‘shish, matritsani songa ko‘paytirish, matritsalarda elementar almashirishlar bajarish, matritsani transponirlash, teskari matritsani topish. Teskari matritsani topishda matritsaning determinanti tushunchasi talab etiladi. Matritsa determinantini aniqlash uchun o‘rin almashtirishlar va ulardagi inversiyalar sonini bilish talab etiladi.

O‘lchamlari teng bo‘lgan A va B matritsalar berilgan bo‘lsin. Aytaylik, $[A] = [B] = m \times n$ bo‘lsin. U holda ularni quyidagicha yozish mumkin

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad \mathrm{B}= \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}.$$

A va B matritsalar faqat va faqat ularning barcha mos elementlari teng bo‘lgandagina **teng** deyiladi. Demak,

$$A = B \Leftrightarrow (a_{ij} = b_{ij},\ i = 1, 2, \dots, m,\ j = 1, 2, \dots, n).$$

A va B matritsalarning **yig‘indisi** deb, ularning barcha mos elementlarining yig‘indisidan tashkil topgan

$$\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}$$

matritsaga aytiladi va $A+B$ kabi belgilanadi. Demak,

$$A+B=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}.$$

Xuddi shuningdek, A va B matritsalarning **ayirmasini** topish mumkin

$$A-B=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \dots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \dots & a_{2n}-b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \dots & a_{mn}-b_{mn} \end{pmatrix}.$$

A matritsaning λ (“lyambda” deb o‘qiladi, grek harfi) **songa ko‘paytmasi** deb,

$$\lambda A=\begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{pmatrix}$$

matritsaga aytiladi.

O‘lchamlari $[A]=m\times n$ bo‘lgan

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

va $[B]=n\times k$ bo‘lgan

$$B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{pmatrix}$$

matritsalar berilgan bo‘lsin. U holda bu matritsalarning $A\cdot B$ **ko‘paytmasi**

$$A\cdot B=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\cdot\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{pmatrix}=$$

$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j}b_{j1} & \sum_{j=1}^{n} a_{1j}b_{j2} & \cdots & \sum_{j=1}^{n} a_{1j}b_{jk} \\ \sum_{j=1}^{n} a_{2j}b_{j1} & \sum_{j=1}^{n} a_{2j}b_{j2} & \cdots & \sum_{j=1}^{n} a_{2j}b_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} a_{mj}b_{j1} & \sum_{j=1}^{n} a_{mj}b_{j2} & \cdots & \sum_{j=1}^{n} a_{mj}b_{jk} \end{pmatrix}$$

kabi topiladi. Boshqacha aytganda

$$A \cdot B = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & a_{m2} & \cdots & c_{mk} \end{pmatrix}$$

ko‘paytmaning mos elementlari

$$c_{pq} = \sum_{j=1}^{n} a_{pj}b_{jq} \ ,$$

tengliklardan topiladi. Bu erda $p = 1, 2, \dots, m$, $q = 1, 2, \dots, k$. Demak, bu matritsalarning $A \cdot B$ ko‘paytmasining o‘lchami $[A \cdot B] = m \times k$ ekan.

Topshiriq. a) O‘lchamlari $[A] = 2 \times 3$ va $[B] = 3 \times 4$ bo‘lgan A va B matritsalar uchun ularning $A \cdot B$ ko‘paytmasini yozing. $B \cdot A$ ko‘paytmani aniqlash mumkinmi?

b) O‘lchamlari $[A] = 4 \times 3$ va $[B] = 3 \times 2$ bo‘lgan A va B matritsalar uchun ularning $A \cdot B$ ko‘paytmasini yozing. $B \cdot A$ ko‘paytmani aniqlash mumkinmi?

2.3. Matritsalardagi elementar almashtirishlar

Matritsalardagi **elementar almashtirishlar** deb quyidagi amallarga aytiladi:

(e1) Matritsa satrlarining o‘rinlarini almashtirish (yoki ustunlarining o‘rinlarini almashtirish);

(e2) Matritsaning biror satrini noldan farqli songa ko‘paytirish (yoki biror ustunini noldan farqli songa ko‘paytirish);

(e3) Matritsaning biror satri elementlariga boshqa biror satri mos elementlarini biror tayinli songa ko‘paytirib qo‘shish (yoki biror ustuni elementlariga boshqa biror ustuni mos elementlarini biror tayinli songa ko‘paytirib qo‘shish).

(e4) Matritsada nol-satrlar va nol-ustunlar bo‘lsa, ularni o‘chirish.

Elementar almashtirishlar natijasida hosil qilingan matritsalar **ekvivalent matritsalar** deyiladi. Agar A va B ekvivalent matritsalar bo‘lsa, $A \sim B$ belgilash ishlatiladi.

Demak,

(e1) xossani quyidagicha ifodalash mumkin

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \sim \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

yoki

$$\begin{pmatrix} \cdots & a_{1l} & \cdots & a_{1j} & \cdots \\ \cdots & a_{2l} & \cdots & a_{2j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{nl} & \cdots & a_{nj} & \cdots \end{pmatrix} \sim \begin{pmatrix} \cdots & a_{1j} & \cdots & a_{1l} & \cdots \\ \cdots & a_{2j} & \cdots & a_{2l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{nj} & \cdots & a_{nl} & \cdots \end{pmatrix};$$

(e2) xossani quyidagicha ifodalash mumkin

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{i\,j} & \cdots & a_{i\,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda a_{i1} & \lambda a_{i2} & \cdots & \lambda a_{i\,j} & \cdots & \lambda a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}, \quad \lambda \neq 0,$$

yoki

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{i\,j} & \cdots & a_{i\,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & \cdots & \lambda a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \lambda a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & \lambda a_{i\,j} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & \lambda a_{mj} & \cdots & a_{mn} \end{pmatrix};$$

(e3) xossani quyidagicha ifodalash mumkin

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \sim \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{i1} + a_{k1} & \lambda a_{i2} + a_{k1} & \cdots & \lambda a_{in} + a_{k1} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

yoki

$$\begin{pmatrix} \cdots & a_{1l} & \cdots & a_{1j} & \cdots \\ \cdots & a_{2l} & \cdots & a_{2j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{nl} & \cdots & a_{nj} & \cdots \end{pmatrix} \sim \begin{pmatrix} \cdots & \lambda a_{1j} + a_{1l} & \cdots & a_{1j} & \cdots \\ \cdots & \lambda a_{2j} + a_{2l} & \cdots & a_{2j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \lambda a_{nj} + a_{nl} & \cdots & a_{nj} & \cdots \end{pmatrix}.$$

Shu o‘rinda matritsalar tengligi va matritsalarning ekvivalentligi turlicha tushunchalar ekanligini ta’kidlab o‘tamiz (misollarda tekshirib, ishonch hosil qiling).

Odatda elementar almashtirishlardan matritsani

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{i\,j} & \cdots & a_{i\,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

ko‘rinishdan **pog‘onali shakl** deb ataluvchi ushbu

$$\begin{pmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1(j-1)}^* & a_{1j}^* & \cdots & a_{1n}^* \\ 0 & a_{22}^* & \cdots & a_{2(j-1)}^* & a_{2j}^* & \cdots & a_{2n}^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{ij}^* & \cdots & a_{in}^* \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \sim \begin{pmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1(j-1)}^* & a_{1j}^* & \cdots & a_{1n}^* \\ 0 & a_{22}^* & \cdots & a_{2(j-1)}^* & a_{2j}^* & \cdots & a_{2n}^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{ij}^* & \cdots & a_{in}^* \end{pmatrix}$$

ko‘rinishga keltirishda foydalanishadi.

2.4. Transponirlangan matritsa

Teskari matritsani topishda transponirlangan matritsa ishlatiladi.

Berilgan satrlarini ustunlar kabi, ustunlarini satrlar kabi yozishdan hosil bo‘lgan matritsa **transponirlangan matritsa** deyiladi. A matritsaning transponirlangan matritsasi $A^{\perp}$ kabi yoziladi. Demak,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ matritsaga } A^{\perp} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix} \text{ matritsa}$$

transponirlangan matritsa bo‘ladi.

Ravshanki, $[A] = m \times n$ bo‘lsa, u holda $[A^{\perp}] = n \times m$ bo‘ladi. Demak, kvadrat matritsalarda $[A] = [A^{\perp}]$ ekan. Undan tashqari, E birlik maritsa uchun $E = E^{\perp}$ tenglik bajarilishi ko‘rinib turibdi.

$A^{\perp} = A$ tenglikni qanoatlantiruvchi A matritsa simmetrik matritsa deyiladi.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

matritsa simmetrik bo‘lishi uchun $a_{ij} = a_{ji}$ tengliklar bajarilishi zarur va yetarli, $i = 1, 2, \dots, m; j = 1, 2, \dots, m$.

$A^{\perp} = -A$ tenglikni qanoatlantiruvchi A matritsa kososimmetrik matritsa deyiladi. Kososimmetrik matritsa $a_{ij} = -a_{ji}$ bo‘ladi, $i = 1, 2, \dots, m; j = 1, 2, \dots, m$.

2.5. O‘rin almashtirishlar va ulardagi inversiyalar soni

Kvadrat matritsaning determinatini aniqlash uchun o‘rin almashtirishlar tushunchasini eslatib o‘tamiz.

Tartibi n bo‘lgan **o‘rin almashtirish** deb, n ta sonning tartiblangan to‘plamiga aytiladi. Bu tartiblangan to‘plamda elementlar o‘rni o‘zgarsa, boshqa o‘rin almashtirish hosil bo‘ladi. Masalan, $\{1, 2, 3, 4\}$ to‘plamda $\langle 1, 2, 3, 4 \rangle$ bitta o‘rin almashtirish, $\langle 1, 2, 4, 3 \rangle$ boshqa o‘rin almashtirish bo‘ladi.

Aslida o‘rin almashtirish degani chekli to‘plamni o‘zini-o‘ziga o‘zaro bir qiymatli akslantirish degani. Masalan, $j: \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ o‘zaro bir qiymatli akslantirish $j(1) = 4$, $j(2) = 3$, $j(3) = 1$, $j(4) = 2$ qoida bilan aniqlangan bo‘lsa, u

$\langle 4,3,1,2\rangle$ o‘rin almashtirishni beradi. Bu holatni quyidagi ko‘rinishda yozish mumkin

$$j=\begin{Bmatrix} 1 & 2 & 3 & 4 \\ j(1) & j(2) & j(3) & j(4)\end{Bmatrix}=\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2\end{Bmatrix}.$$

n elementli to‘plamdagi $j\colon\{1,2,3,\dots,n\}\to\{1,2,3,\dots,\ n\}$ o‘rin almashtirishni

$$j=\begin{Bmatrix} 1 & 2 & \dots & n \\ j(1) & j(2) & \dots & j(n)\end{Bmatrix}$$

kabi yozish mumkin.

$\{1,\ 2\}$ to‘plamdagi o‘rin almatirishshlar soni 2 ga teng:

$$\begin{Bmatrix} 1 & 2 \\ 1 & 2\end{Bmatrix},\quad \begin{Bmatrix} 1 & 2 \\ 2 & 1\end{Bmatrix}.$$

$\{1,\ 2,\ 3\}$ to‘plamdagi o‘rin almatirishlar 6 ga teng:

$$\begin{Bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1\end{Bmatrix}.$$

$\{1,\ 2,\ 3,\ 4\}$ to‘plamda o‘rin almatirishlar 24 ga teng:

$$\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2\end{Bmatrix};$$
$$\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1\end{Bmatrix};$$
$$\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2\end{Bmatrix};$$
$$\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3\end{Bmatrix},\ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2\end{Bmatrix}.$$

n ta elementdan iborat $\{1,\ 2,\ 3,\dots,\ n\}$ to‘plamda

$$n!=1\cdot 2\cdot 3\cdot\ldots\cdot n$$

ta o‘rin almatirishlar mavjud ($n!$ yozuv “en-faktorial” deb o‘qiladi).

Yuqoridagi misollarda $2=1\cdot 2=2!$ (ikki-faktorial), $6=1\cdot 2\cdot 3=3!$ (uch-faktorial), $24=1\cdot 2\cdot 3\cdot 4=4!$ (to‘rt-faktorial).

Ushbu

$$j=\begin{Bmatrix} 1 & 2 & \dots & n \\ j(1) & j(2) & \dots & j(n)\end{Bmatrix}$$

o‘rin almashtirishni qaraylik. Agarda $l,\ m\in\{1,2,\dots,n\}$ sonlar jufti uchun $l<m$ va $j(l)>j(m)$ bo‘lsa, u holda (l,m) juftlik mazkur o‘rin almashtirishda **inversiya** hosil qiladi deyiladi.

Yuqoridagi

$$j=\begin{Bmatrix} 1 & 2 & 3 & 4 \\ j(1) & j(2) & j(3) & j(4)\end{Bmatrix}=\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2\end{Bmatrix}.$$

o‘rin almashtirishda $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$ juftliklar inversiya hosil qiladi. Demak, bu o‘rin almashtirishda jami 5 ta inversiya mavjud.

n ta elementdan iborat $\{1,\ 2,\ 3,\dots,\ n\}$ to‘plamda $n!$ ta o‘rin almatirishlar mavjudligini eslatgan edik. j_k orqali k inchi o‘rin almashtirishni belgilaylik, $k=1,2,\dots,n!$ Bu j_k o‘rin almashtirishlardagi inversiyalar soni N_k bo‘lsin. Demak,

$$j_k=\begin{Bmatrix} 1 & 2 & \dots & n \\ j_k(1) & j_k(2) & \dots & j_k(n)\end{Bmatrix},$$
$$N_k=|\{(l,m)\colon l<m,\qquad j_k(l)>j_k(m)\}|.$$

Shu o‘rinda chekli A to‘plamning elementlarining sonini $|A|$ orqali belgilanishini eslatib o‘tamiz. Masalan, $A = \{a,\ b,\ c,\ 7,\ \pi,\ \sin 15^0\}$ to‘plam elementlari soni $|A| = 6$.

2.6. Kvadrat matritsaning determinanti

Endi

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

kvadrat matritsaning determinati tushunchasini kiritish mumkin. A matritsaning **determinanti** deb

$$\sum_{k=1}^{n!} (-1)^{N_k} \cdot a_{1j_k(1)} \cdot a_{2j_k(2)} \cdot \dots \cdot a_{nj_k(n)}$$

yig‘indiga aytiladi va $\det A$ kabi belgilashadi.

Demak,

$$\det A = \sum_{k=1}^{n!} (-1)^{N_k} \cdot a_{1j_k(1)} \cdot a_{2j_k(2)} \cdot \dots \cdot a_{nj_k(n)} \ .$$

Bu formulani tavsiflaymiz. O‘lchamlari $n \times n$ bo‘lgan A kvadrat matritsaning **demerminanti** deb $n!$ ta qo‘shiluvchidan iborat quyidagi talablarni qanoatlantiruvchi yig‘indiga aytiladi:

1) Har bir qo‘shiluvchi matritsaning n ta elementi ko‘paytmasidan iborat bo‘lib, unda har bir satrdan va har bir ustundan (ular kesishmasidagi yagona) bittadan element ishtirok etadi.

2) k-yig‘indidagi ko‘paytma undagi ko‘paytuvchilar (ya’ni A matritsa elementlari) satrlar tartibi bilan yozilgan bo‘lsin. U holda j_k o‘rin almashtirishlardagi inversiyalar soni N_k toq bo‘lsa, k-yig‘indi oldida (-1) koeffitsient hosil bo‘ladi, N_k juft bo‘lsa, k-yig‘indi oldidagi ishora musbat bo‘ladi.

Bir xil tartibli A va B kvadrat matritsalarning $A \cdot B$ ko‘paytmasining determinanti bu matritsalar determinantlarining ko‘paytmasiga teng:

$$\det(A \cdot B) = \det A \cdot \det B \ .$$

A kvadrat matritsaning va uning transponirlashdan hosil qilingan $A^{\perp}$ matritsaning determinantlari teng bo‘ladi:

$$\det A = \det A^{\perp} \ .$$

2.6.1. Ikkinchi tartibli matritsa determinantini topish

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

matritsaning determinantini topamiz. Bu matritsaning o‘lchami 2×2, ya’ni $n = 2$. O‘rin almashtirishlar

$$j_1 = \begin{Bmatrix} 1 & 2 \\ 1 & 2 \end{Bmatrix},\ j_2 = \begin{Bmatrix} 1 & 2 \\ 2 & 1 \end{Bmatrix}.$$

Boshqa o‘rin almashtirishlar yo‘q. j_1 o‘rin almashtirishda inversiya mavjud emas, ya’ni $N_1 = 0$. j_2 o‘rin almashtirishda esa bitta inversiya mavjud $N_2 = 1$.

Aytilganlarni jadval ko‘rinishda yozaylik.

k	$\{1,2\}$ to‘plamdagi barcha o‘rin almashtirishlar	j_k o‘rin almashtirishdagi inversiyalar soni N_k	Mos $(-1)^{N_k} \cdot a_{1j_k(1)} \cdot a_{2j_k(2)}$ ko‘paytma
1	$j_1 = \begin{Bmatrix} 1 & 2 \\ 1 & 2 \end{Bmatrix}$	0	$a_{11} \cdot a_{22}$
2	$j_2 = \begin{Bmatrix} 1 & 2 \\ 2 & 1 \end{Bmatrix}$	1 ta $\left(j_2(1) > j_1(2)\right)$	$-a_{12} \cdot a_{21}$

Demak,

$$\det A = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \sum_{k=1}^{2!} (-1)^{N_k} \cdot a_{1j_k(1)} \cdot a_{2j_k(2)} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}\ ,$$

ya’ni

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}\ .$$

2.6.2. Uchinchi tartibli matritsa determinantini topish

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

matritsaning determinantini topamiz. Bu matritsaning o‘lchami 3×3, ya’ni $n = 3$. Ushbu jadvaldan foydalanamiz.

k	$\{1,2,3\}$ to‘plamdagi barcha o‘rin almashtirishlar	j_k o‘rin almashtirishdagi inversiyalar soni N_k	Mos $(-1)^{N_k} \cdot a_{1j_k(1)} \cdot a_{2j_k(2)} \cdot a_{3j_k(3)}$ ko‘paytma
1	$j_1 = \begin{Bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{Bmatrix}$	0	$a_{11} \cdot a_{22} \cdot a_{33}$
2	$j_2 = \begin{Bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{Bmatrix}$	1 ta $(j_2(2) > j_2(3))$	$-a_{11} \cdot a_{23} \cdot a_{32}$
3	$j_3 = \begin{Bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{Bmatrix}$	1 ta $(j_3(1) > j_3(2))$	$-a_{12} \cdot a_{23} \cdot a_{32}$
4	$j_4 = \begin{Bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{Bmatrix}$	2 ta $\left(j_4(1) > j_4(3)\ va\ j_4(2) > j_4(3)\right)$	$a_{12} \cdot a_{21} \cdot a_{33}$
5	$j_5 = \begin{Bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{Bmatrix}$	2 ta $\left(j_5(1) > j_5(3)\ va\ j_5(1) > j_5(3)\right)$	$a_{13} \cdot a_{21} \cdot a_{32}$
6	$j_6 = \begin{Bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{Bmatrix}$	3 ta $(j_6(1) > j_6(2),\ j_6(1) > j_6(3)\ va\ j_6(2) > j_6(3))$	$-a_{13} \cdot a_{22} \cdot a_{31}$

Bundan

$$\det A = \sum_{k=1}^{3!} (-1)^{N_k} \cdot a_{1j_k(1)} \cdot a_{2j_k(2)} \cdot a_{3j_k(3)} =$$

$$= a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32} - a_{12} \cdot a_{23} \cdot a_{32} + a_{12} \cdot a_{21} \cdot a_{33} + a_{13} \cdot a_{21} \cdot a_{32} - a_{13} \cdot a_{22} \cdot a_{31},$$

ya’ni

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32} - a_{12} \cdot a_{23} \cdot a_{32} + a_{12} \cdot a_{21} \cdot a_{33} + a_{13} \cdot a_{21} \cdot a_{32} - a_{13} \cdot a_{22} \cdot a_{31}.$$

2.6.3. To‘rtinchi tartibli matritsa determinantini topish

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

matritsaning determinantini topamiz. Bu matritsaning o‘lchami 4×4, ya’ni $n = 4$. Ushbu jadvaldan foydalanamiz.

k	$\{1,2,3,4\}$ to‘plamdagi barcha o‘rin almashtirishlar	j_k o‘rin almashtirishdagi inversiyalar soni N_k	Mos $(-1)^{N_k} \cdot a_{1j_k(1)} \cdot a_{2j_k(2)} \cdot a_{3j_k(3)} \cdot a_{4j_k(4)}$ ko‘paytma
1	$j_1 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{Bmatrix}$	0	$a_{11} \cdot a_{22} \cdot a_{33} \cdot a_{44}$
2	$j_2 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{Bmatrix}$	1 ta $(j_2(3) > j_2(4))$	$-a_{11} \cdot a_{22} \cdot a_{34} \cdot a_{43}$
3	$j_3 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{Bmatrix}$	1 ta $(j_3(2) > j_3(3))$	$-a_{11} \cdot a_{23} \cdot a_{32} \cdot a_{44}$
4	$j_4 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{Bmatrix}$	2 ta $\left(j_4(2) > j_4(4)\ va\ j_4(3) > j_4(4)\right)$	$a_{11} \cdot a_{23} \cdot a_{34} \cdot a_{42}$
5	$j_5 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{Bmatrix}$	2 ta $\left(j_5(2) > j_5(3)\ va\ j_5(2) > j_5(4)\right)$	$a_{11} \cdot a_{24} \cdot a_{32} \cdot a_{43}$
6	$j_6 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{Bmatrix}$	3 ta $(j_6(2) > j_6(3),\ j_6(2) > j_6(4)\ va\ j_6(3) > j_6(4))$	$-a_{11} \cdot a_{24} \cdot a_{33} \cdot a_{42}$
7	$j_7 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{Bmatrix}$	1 ta $(j_7(1) > j_7(2))$	$-a_{12} \cdot a_{21} \cdot a_{34} \cdot a_{44}$
8	$j_8 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{Bmatrix}$	2 ta $\left(j_8(1) > j_8(2)\ va\ j_8(3) > j_8(4)\right)$	$a_{12} \cdot a_{21} \cdot a_{34} \cdot a_{43}$
9	$j_9 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{Bmatrix}$	2 ta $\left(j_9(1) > j_9(3)\ va\ j_9(2) > j_9(3)\right)$	$a_{12} \cdot a_{23} \cdot a_{31} \cdot a_{44}$
10	$j_{10} = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{Bmatrix}$	3 ta $(j_{10}(1) > j_{10}(4),\ j_{10}(2) > j_{10}(4)\ va\ j_{10}(3) > j_{10}(4))$	$-a_{12} \cdot a_{23} \cdot a_{34} \cdot a_{41}$
11	$j_{11} = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{Bmatrix}$	3 ta $(j_{11}(1) > j_{11}(2),\ j_{11}(2) > j_{11}(3)\ va\ j_{11}(2) > j_{11}(4))$	$-a_{12} \cdot a_{24} \cdot a_{31} \cdot a_{43}$
12	$j_{12} = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{Bmatrix}$	4 ta $(j_{12}(1) > j_{12}(4),\ j_{12}(2) > j_{12}(4),\ j_{12}(3) > j_{12}(4)\ va\ j_{12}(2) > j_{12}(3))$	$a_{12} \cdot a_{24} \cdot a_{33} \cdot a_{41}$

13	$j_{13}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 3 & 2 & 1 & 4\end{Bmatrix}$	3 ta $(j_{13}(1)>j_{13}(2),\ j_{13}(1)>j_{13}(3)\ va\ j_{13}(2)>j_{13}(3))$	$-a_{13}\cdot a_{22}\cdot a_{31}\cdot a_{44}$
14	$j_{14}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 3 & 2 & 4 & 1\end{Bmatrix}$	4 ta $(j_{14}(1)>j_{14}(2),\ j_{14}(1)>j_{14}(4),\ j_{14}(2)>j_{14}(4)\ va\ j_{14}(3)>j_{14}(4))$	$a_{13}\cdot a_{22}\cdot a_{34}\cdot a_{41}$
15	$j_{15}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 2 & 4\end{Bmatrix}$	2 ta $(j_{15}(1)>j_{15}(2)\ va\ j_{15}(1)>j_{15}(3))$	$a_{13}\cdot a_{21}\cdot a_{32}\cdot a_{44}$
16	$j_{16}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 4 & 2\end{Bmatrix}$	3 ta $(j_{16}(1)>j_{16}(2),\ j_{16}(1)>j_{16}(4)\ va\ j_{16}(3)>j_{16}(4))$	$-a_{13}\cdot a_{21}\cdot a_{34}\cdot a_{42}$
17	$j_{17}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 2 & 1\end{Bmatrix}$	5 ta $(j_{17}(1)>j_{17}(3),\ j_{17}(1)>j_{17}(4),\ j_{17}(2)>j_{17}(3),\ j_{17}(2)>j_{17}(4)$ va $j_{17}(3)>j_{17}(4))$	$-a_{13}\cdot a_{24}\cdot a_{32}\cdot a_{41}$
18	$j_{18}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{Bmatrix}$	4 ta $(j_{18}(1)>j_{18}(3),\ j_{18}(1)>j_{18}(4),\ j_{18}(2)>j_{18}(3)\ va\ j_{18}(2)>j_{18}(4))$	$a_{13}\cdot a_{24}\cdot a_{31}\cdot a_{42}$
19	$j_{19}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 4 & 2 & 3 & 1\end{Bmatrix}$	5 ta $(j_{19}(1)>j_{19}(2),\ j_{19}(1)>j_{19}(3),\ j_{19}(1)>j_{19}(4),\ j_{19}(2)>j_{19}(4)$ va $j_{19}(3)>j_{19}(4))$	$-a_{14}\cdot a_{22}\cdot a_{33}\cdot a_{41}$
20	$j_{20}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 4 & 2 & 1 & 3\end{Bmatrix}$	4 ta $(j_{20}(1)>j_{20}(2),\ j_{20}(1)>j_{20}(3),\ j_{20}(1)>j_{20}(4)\ va\ j_{20}(2)>j_{20}(3))$	$a_{14}\cdot a_{22}\cdot a_{31}\cdot a_{43}$
21	$j_{21}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{Bmatrix}$	6 ta $(j_{21}(1)>j_{21}(2),\ j_{21}(1)>j_{21}(3),\ j_{21}(1)>j_{21}(4),\ j_{21}(2)>j_{21}(3),$ $j_{21}(2)>j_{21}(4)$ va $j_{21}(3)>j_{21}(4))$	$a_{14}\cdot a_{23}\cdot a_{32}\cdot a_{41}$
22	$j_{22}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 1 & 2\end{Bmatrix}$	5 ta $(j_{22}(1)>j_{22}(2),\ j_{22}(1)>j_{22}(3),\ j_{22}(1)>j_{22}(4),\ j_{22}(2)>j_{22}(3)$ va $j_{22}(2)>j_{22}(4))$	$-a_{14}\cdot a_{23}\cdot a_{31}\cdot a_{42}$
23	$j_{23}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 2 & 3\end{Bmatrix}$	3 ta $(j_{23}(1)>j_{23}(2),\ j_{23}(1)>j_{23}(3)\ va\ j_{23}(1)>j_{23}(4))$	$-a_{14}\cdot a_{21}\cdot a_{32}\cdot a_{43}$
24	$j_{24}=\begin{Bmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 3 & 2\end{Bmatrix}$	4 ta $(j_{24}(1)>j_{24}(2),\ j_{24}(1)>j_{24}(3),\ j_{24}(1)>j_{24}(4)\ va\ j_{24}(3)>j_{24}(4))$	$a_{14}\cdot a_{21}\cdot a_{33}\cdot a_{42}$

Bundan

$$\det A=\det\begin{pmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}=\sum_{k=1}^{4!}(-1)^{N_k}\cdot a_{1j_k(1)}\cdot a_{2j_k(2)}\cdot a_{3j_k(3)}\cdot a_{4j_k(4)}=$$

$$\begin{aligned}&=a_{11}\cdot a_{22}\cdot a_{33}\cdot a_{44}-a_{11}\cdot a_{22}\cdot a_{34}\cdot a_{43}-a_{11}\cdot a_{23}\cdot a_{32}\cdot a_{44}+a_{11}\cdot a_{23}\cdot a_{34}\cdot a_{42}+a_{11}\cdot a_{24}\cdot a_{32}\cdot a_{43}-a_{11}\cdot a_{24}\cdot a_{33}\cdot a_{42}-\\&-a_{12}\cdot a_{21}\cdot a_{34}\cdot a_{44}+a_{12}\cdot a_{21}\cdot a_{34}\cdot a_{43}+a_{12}\cdot a_{23}\cdot a_{31}\cdot a_{44}-a_{12}\cdot a_{23}\cdot a_{34}\cdot a_{41}-a_{12}\cdot a_{24}\cdot a_{31}\cdot a_{43}+a_{12}\cdot a_{24}\cdot a_{33}\cdot a_{41}+\\&-a_{13}\cdot a_{22}\cdot a_{31}\cdot a_{44}+a_{13}\cdot a_{22}\cdot a_{34}\cdot a_{41}+a_{13}\cdot a_{21}\cdot a_{32}\cdot a_{44}-a_{13}\cdot a_{21}\cdot a_{34}\cdot a_{42}-a_{13}\cdot a_{24}\cdot a_{32}\cdot a_{41}+a_{13}\cdot a_{24}\cdot a_{31}\cdot a_{42}-\\&-a_{14}\cdot a_{22}\cdot a_{33}\cdot a_{41}+a_{14}\cdot a_{22}\cdot a_{31}\cdot a_{43}+a_{14}\cdot a_{23}\cdot a_{32}\cdot a_{41}-a_{14}\cdot a_{23}\cdot a_{31}\cdot a_{42}-a_{14}\cdot a_{21}\cdot a_{32}\cdot a_{43}+a_{14}\cdot a_{21}\cdot a_{33}\cdot a_{42}\ .\end{aligned}$$

2.7. Teskari matritsa

A va B kvadrat matritsalarning o‘lchamlari teng, ya’ni $[A] = [B] = n \times n$ bo‘lsin. Agar $A \cdot B = E$, ya’ni

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

bo‘lsa, u holda B matritsa A matritsaga teskari matritsa deyiladi va A^{-1} kabi belgilanadi.

A matritsaning teskarisi mavjud bo‘lishi uchun uning determinanti noldan farqli, ya’ni $\det A \neq 0$ bo‘lishi zarur va etarli.

Determinati noldan farqli A matritsaning A^{-1} teskari matritsasini ushbu

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

formula orqali topish mumkin. Bu erda A_{ij} ifoda a_{ij} elementning algebraik to‘ldirmasi, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$.

a_{ij} elementning A_{ij} algebraik to‘ldirmasi quyidagicha topiladi

$$A_{ij} = (-1)^{i+j} \cdot \det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1(j-1)} & a_{1(j+1)} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(j-1)} & a_{2(j+1)} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(j-1)} & a_{n(j+1)} & \dots & a_{nn} \end{pmatrix}.$$

Demak, a_{ij} elementning A_{ij} **algebraik to‘ldirmasi** A matritsadan a_{ij} element joylashgan satrni (i-satrni) va ustunni (j-ustunni) olib tashlash natijasida hosil bo‘lgan $(n-1) \times (n-1)$ o‘lchamli matritsa determininantining $(-1)^{i+j}$ koeffitsientga ko‘paytmasidan iborat ekan.

2.8. Matritsa rangi

Ushbu

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

matritsa berilgan bo‘lsin. Elementlari A matritsaning elementlaridan tuzilgan ushbu

$$\begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_k} \\ a_{i_2 j_1} & a_{i_2 j_2} & \cdots & a_{i_2 j_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k j_1} & a_{i_k j_2} & \cdots & a_{i_k j_k} \end{vmatrix}$$

determinant A matritsaning k inchi tartibli minori deyiladi. Ravshanki, bu determinant A matritsaning k ta i_1-, i_2-, …, i_k-satrlarining har biri bilan k ta j_1-, j_2-, …, j_k-ustunlari kesishmalaridagi elementlardan tashkil topgan. Masalan, $n \geq 7$ bo‘lganda

$$\begin{vmatrix} a_{32} & a_{35} \\ a_{72} & a_{75} \end{vmatrix}$$

determinant A matritsaning 2 tartibli minori bo‘ladi. Yoki

$$\begin{vmatrix} a_{23} & a_{25} & a_{27} \\ a_{43} & a_{45} & a_{47} \\ a_{53} & a_{55} & a_{57} \end{vmatrix}$$

determinant A matritsaning 3 tartibli minori bo‘ladi.

Ushbu

$$\begin{vmatrix} a_{23} & a_{15} & a_{26} \\ a_{44} & a_{36} & a_{13} \\ a_{57} & a_{65} & a_{17} \end{vmatrix}$$

determinant A matritsaning minori bo‘la olmaydi (**nega?**).

Savol: O‘lchamlari $m \times n$ bo‘lgan matritsaning k-tartibli barcha minorlari soni nechta?

Javob: $C_m^k \cdot C_n^k$ ta (isbotlang). Bu erda

$$C_p^q = \frac{p!}{q!\,(p-q)!}\ .$$

Matritsaning rangi tushunchasi juda muhim hisoblanadi. Bu tushunchaning ta’rifini keltiramiz

Ta’rif (matritsa rangi ta’rifi). A matritsada:

($r1$) tartibi r bo‘lgan noldan farqli minor mavjud;

($r2$) tartibi $(r+1)$ bo‘lgan minor mavjud bo‘lsa, ularning har biri nolga teng

shartlarni qanoatlantiruvchi r soni A matritsaning **rangi** deyiladi va $rank\ A$ orqali belgilanadi.

Boshqacha aytganda, A matritsaning noldan farqli minorlarining eng yuqori tartibi A matritsaning rangi deyiladi.

Ta’rifdan o‘lchamlari $[A] = m \times n$ bo‘lgan A matritsa uchun

$$0 \leq rang\ A \leq \min\{m, n\}$$

bo‘lishi kelib chiqadi.

n-tartibli birlik E matritsaning rangi n ga teng.

Har qanday kososimmetrik matritsaning rangi juft son bo‘ladi.

Ta’rifdan quyidagi **xossa** kelib chiqadi: biror k son uchun matritsaning k-tartibli barcha minorlari nolga tengligidan matritsaning tartibi $(k+1)$ va undan katta bo‘lgan har bir minori (agar mavjud bo‘lsa) nolga teng bo‘ladi.

$rang\ A = 0$ bo‘lishi uchun A matritsaning barcha elementlari nol bo‘lishi kerak. Agarda A matritsaning birorta satrida yoki ustunida hech bo‘lmaganda bitta noldan farqli element mavjud bo‘lsa, bu matritsa uchun $rang\ A \geq 1$ bo‘ladi.

Rangi r bo‘lgan A matritsa berilgan bo‘lsin. Uning tartibi r bo‘lgan ixtiyoriy minori A matritsaning **bazis minori** deyiladi. Bazis minorni hosil qiluvchi A

matritsaning satrlari va ustunlari A matritsaning mos ravishda bazis satrlari va bazis ustunlari deb yuritiladi.

Asosiy xossa. A matritsaning har bir satri bazis satrlarning chiziqli kombinatsiyasi bo‘ladi. SHunga o‘xshash A matritsaning har bir ustuni bazis ustunlarning chiziqli kombinatsiyasi bo‘ladi.

Endi asosiy xossadan kelib chiqadigan quyidagi xossalarni sanab o‘tamiz.

1^0. Rangi r bo‘lgan matritsaning har bir $(r+1)$ ta satri (yoki $(r+1)$ ta ustuni) chiziqli bog‘liq bo‘ladi.

2^0. Agar matritsaning rangi uning satrlari sonidan kichik bo‘lsa, u holda uning satrlari chiziqli bog‘liq bo‘ladi. Agar matritsaning rangi uning satrlari soniga teng bo‘lsa, u holda uning satrlari chiziqli bog‘liqmas bo‘ladi.

Xuddi shuningdek, Agar matritsaning rangi uning ustunlari sonidan kichik bo‘lsa, u holda uning ustunlari chiziqli bog‘liq bo‘ladi. Agar matritsaning rangi uning ustunlari soniga teng bo‘lsa, u holda uning ustunlari chiziqli bog‘liqmas bo‘ladi.

3^0. Matritsaning determinanti faqat va faqat matritsaning barcha satrlari (yoki ustunlari) chiziqli bog‘liq bo‘lsagina nolga teng bo‘ladi.

4^0. Elementar almashtirishlar bajarilganda matritsaning rangi o‘zgarmaydi.

Ravshanki, 4^0 xossani quyidagicha yozish mumkin.

$4^{0\prime}$. Ekvivalent matritsalarning ranglari teng, ya’ni

$$A \sim B \Longrightarrow rang\ A = rang\ B\ .$$

Quyidagi ikkita misolda matritsa rangini bevosita ta’rifdan topishga qaraganda, elementar almashtirishlar bajarib topish qulay ekanligini ko‘rish mumkin.

2.8.1.-misol.

$$\begin{pmatrix} 2 & -1 & 3 & -2 & 4 \\ 4 & -2 & 5 & 1 & 7 \\ 2 & -1 & 1 & 8 & 2 \end{pmatrix}$$

matritsa rangini minorlar yordamida toping.

Yechish. Berilgan matritsani A deylik. A matritsaning ikkinchi tartibli minorlarini hisoblaymiz (ularning soni $C_3^2 \cdot C_5^2 = 3 \cdot 10 = 30$ ta). Ishni birinchi noldan farqli minor topilguncha davom ettiramiz.

$$\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = -4 + 4 = 0, \quad \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2 \neq 0\ .$$

Demak, $rang\ A \geq 2$ ekan.

Endi A matritsaning uchinchi tartibli minorlarini hisoblaymiz (ularning soni $C_3^3 \cdot C_5^3 = 1 \cdot 10 = 10$ ta). Ishni birinchi noldan farqli minor topilguncha davom ettiramiz.

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -1 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 8 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -1 & 4 \\ 4 & -2 & 7 \\ 2 & -1 & 2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 2 & 3 & -2 \\ 4 & 5 & 1 \\ 2 & 1 & 8 \end{vmatrix} = 80 + 6 - 8 + 20 - 2 - 96 = 0,$$

$$\begin{vmatrix} 2 & 3 & 4 \\ 4 & 5 & 7 \\ 2 & 1 & 2 \end{vmatrix} = 20 + 42 + 16 - 40 - 14 - 24 = 0,$$

$$\begin{vmatrix} 2 & -2 & 4 \\ 4 & 1 & 7 \\ 2 & 8 & 2 \end{vmatrix} = 4 - 28 + 128 - 8 + 16 - 112 = 0,$$

$$\begin{vmatrix} -1 & 3 & -2 \\ -2 & 5 & 1 \\ -1 & 1 & 8 \end{vmatrix} = -40 - 3 + 4 - 10 + 48 + 1 = 0,$$

$$\begin{vmatrix} -1 & 3 & 4 \\ -2 & 5 & 7 \\ -1 & 1 & 2 \end{vmatrix} = -10 - 21 - 8 + 20 + 12 + 7 = 0,$$

$$\begin{vmatrix} -1 & -2 & 4 \\ -2 & 1 & 7 \\ -1 & 8 & 2 \end{vmatrix} = -2 + 14 - 64 + 4 + 56 - 8 = 0,$$

$$\begin{vmatrix} 3 & -2 & 4 \\ 5 & 1 & 7 \\ 1 & 8 & 2 \end{vmatrix} = 6 - 14 + 160 - 4 + 20 - 168 = 0.$$

Berilgan matritsaning barcha 3-tartibli minorlari nol chiqdi. Lekin unda noldan farqli 2-tartibli minori mavjudligini yuqorida ko‘rsatdik. Demak, bu misolda $rang\ A = 2$ ekan.

Savol: Nima uchun bu misolni Yechishda dastlabki uchta 3-tartibli minorlarni hisoblamasdan ularning qiymatinni 0 ga teng deb ketdik?

Javob: Chunki bu uchta 3-tartibli minorlarda 1- va 2-ustunlar proporsional.

2.8.2.-misol.

$$\begin{pmatrix} 5 & -3 & 7 & 13 \\ 10 & -5 & 16 & 29 \\ 15 & -8 & 24 & 44 \\ -5 & 4 & -4 & -8 \end{pmatrix}$$

matritsa rangini elementar almashtirishlar yordamida toping.

Yechish. Berilgan matritsani A orqali belgilaylik. U holda navbati bilan quyidagilarga ega bo‘lamiz

$$A \sim \begin{pmatrix} 5 & -3 & 7 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 5 & -3 & 7 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 5 & -3 & 7 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B\ .$$

Endi B matritsaning noldan farqli uchinchi tartibli minorini ko‘rsatamiz: $\begin{vmatrix} 25 & 31 & 17 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 25 \neq 0$. B matritsada noldan farqli bironta ham 4-tartibli minor mavjud emasligi ko‘rinib turibdi (**asoslang**). Shuning uchun

Javob: $rang\ A = rang\ B = 3$.

Mustahkamlash uchun mashqlar

Matritsalarga oid masalalar

1. Berilgan matritsalar ustida ko‘rsatilgan amallarni bajaring

1) $A = \begin{pmatrix} 1 & -1 & 7 \\ 0 & 2 & 0 \\ -4 & 5 & 3 \end{pmatrix}$, $\quad 3 \cdot A = ?$

2) $A = \begin{pmatrix} 3 & 0 & 7 \\ 7 & 1 & 5 \\ -2 & 0 & 4 \end{pmatrix}$, $\quad -\frac{4}{7} \cdot A = ?$

3) $A = \begin{pmatrix} 1 & 3 & 7 \\ 7 & 1 & 5 \\ -2 & 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 7 \\ 0 & 2 & 0 \\ -4 & 5 & 3 \end{pmatrix}$, $\quad A + B = ?$

4) $A = \begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & 4 & 8 & 12 \\ -2 & -4 & 6 & -10 \\ 7 & 21 & -7 & 56 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 7 \end{pmatrix}$, $\quad 3 \cdot A - 2 \cdot B = ?$

6) $A = \begin{pmatrix} 3 & 2 & -2 & 3 \\ 1 & 1 & 2 & 3 \\ 3 & 2 & -1 & 1 \\ 1 & 1 & 0 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & -2 & 31 \\ 3 & 11 & 2 & 23 \\ 1 & 2 & -1 & 1 \\ 4 & 1 & 5 & 17 \end{pmatrix}$, $\quad -5 \cdot A + 4 \cdot B = ?$

2. Matritsalarning ko‘paytmasini toping

1) $A = \begin{pmatrix} -1 & -1 & 5 & -2 \\ 0 & 1 & 2 & 3 \\ 4 & 8 & -12 & 20 \\ 3 & 7 & 7 & 18 \\ 1 & 3 & -1 & 8 \end{pmatrix}$, $\quad B = \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 7 \end{pmatrix}$, $\quad A \cdot B = ?$

2) $A = \begin{pmatrix} 1 & 3 & 7 \\ 7 & 1 & 5 \\ -2 & 0 & 4 \end{pmatrix}$, $\quad B = \begin{pmatrix} 1 & -1 & 7 \\ 0 & 2 & 0 \\ -4 & 5 & 3 \end{pmatrix}$, $\quad A \cdot B = ?$

3) $A = \begin{pmatrix} 3 & 2 & -2 & 3 \\ 1 & 1 & 2 & 3 \\ 3 & 2 & -1 & 1 \\ 1 & 1 & 0 & 7 \end{pmatrix}$, $\quad B = \begin{pmatrix} 1 & 2 & -2 & 31 \\ 3 & 11 & 2 & 23 \\ 1 & 2 & -1 & 1 \\ 4 & 1 & 5 & 17 \end{pmatrix}$, $\quad A \cdot B = ?$

4) $A = \begin{pmatrix} 3 & 2 & -2 & 3 \\ 1 & 1 & 2 & 3 \\ 3 & 2 & -1 & 1 \\ 1 & 1 & 0 & 7 \end{pmatrix}$, $\quad B = \begin{pmatrix} 1 & 2 & -2 & 31 \\ 3 & 11 & 2 & 23 \\ 1 & 2 & -1 & 1 \\ 4 & 1 & 5 & 17 \end{pmatrix}$, $\quad A \cdot B = ?$

5) $A = \begin{pmatrix} 0 & 1 & 4 & 5 \\ 2 & 0 & 3 & 1 \end{pmatrix}$, $\quad B = \begin{pmatrix} 1 & 2 & -2 & 31 \\ 3 & 11 & 2 & 23 \\ 1 & 2 & -1 & 1 \\ 4 & 1 & 5 & 17 \end{pmatrix}$, $\quad A \cdot B = ?$

6) $A = \begin{pmatrix} 3 & 2 & -2 & 3 \\ 1 & 1 & 2 & 3 \\ 3 & 2 & -1 & 1 \\ 1 & 1 & 0 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \qquad A \cdot B = ?$

3. Matritsaning determinantini toping

1) $\begin{pmatrix} 1 & 2 & -3 & 5 \\ 3 & 7 & -7 & 18 \\ -2 & -2 & 7 & -12 \\ 7 & 14 & -21 & 42 \end{pmatrix}$, 2) $\begin{pmatrix} 1 & 5 & 3 & 11 \\ 0 & 1 & 2 & 3 \\ 0 & 9 & 19 & 25 \\ 0 & 0 & -2 & 5 \end{pmatrix}$, 3) $\begin{pmatrix} 1 & 2 & -3 & 5 \\ -1 & -1 & 5 & -2 \\ 0 & -1 & -1 & -1 \\ 3 & 3 & -15 & 7 \end{pmatrix}$,

4) $\begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & 4 & 8 & 12 \\ -2 & -4 & 6 & -10 \\ 2 & 3 & -1 & 8 \end{pmatrix}$, 5) $\begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & -1 & 1 \\ 2 & 3 & -2 & 10 \end{pmatrix}$, 6) $\begin{pmatrix} 3 & 2 & -2 & 3 \\ 1 & 1 & 2 & 3 \\ 3 & 2 & -1 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}$,

4. Berilgan matritsaga teskari matritsani toping

1) $\begin{pmatrix} 1 & -1 & 7 \\ 0 & 2 & 0 \\ -4 & 5 & 3 \end{pmatrix}$, 2) $\begin{pmatrix} 3 & 0 & 7 \\ 7 & 1 & 5 \\ -2 & 0 & 4 \end{pmatrix}$, 3) $\begin{pmatrix} 1 & 3 & 7 \\ 7 & 1 & 5 \\ -2 & 0 & 4 \end{pmatrix}$,

4) $\begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & 4 & 8 & 12 \\ -2 & -4 & 6 & -10 \\ 7 & 21 & -7 & 56 \end{pmatrix}$, 5) $\begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 7 \end{pmatrix}$, 6) $\begin{pmatrix} 3 & 2 & -2 & 3 \\ 1 & 1 & 2 & 3 \\ 3 & 2 & -1 & 1 \\ 1 & 1 & 0 & 7 \end{pmatrix}$.

5. Berilgan matritsaning rangini minorlar yordamida toping

1) $\begin{pmatrix} 1 & 2 & -3 & 5 \\ 3 & 7 & -7 & 18 \\ -2 & -2 & 7 & -12 \\ 7 & 14 & -21 & 35 \end{pmatrix}$, 2) $\begin{pmatrix} 1 & 5 & 3 & 11 \\ 0 & 1 & 2 & 3 \\ 0 & 9 & 19 & 25 \\ 0 & 0 & -2 & 4 \end{pmatrix}$, 3) $\begin{pmatrix} 1 & 2 & -3 & 5 \\ -1 & -1 & 5 & -2 \\ 0 & -1 & -1 & -1 \\ 3 & 3 & -15 & 6 \end{pmatrix}$,

4) $\begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & 4 & 8 & 12 \\ -2 & -4 & 6 & -10 \\ 7 & 21 & -7 & 56 \end{pmatrix}$, 5) $\begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 7 \end{pmatrix}$, 6) $\begin{pmatrix} 3 & 2 & -2 & 3 \\ 1 & 1 & 2 & 3 \\ 3 & 2 & -1 & 1 \\ 1 & 1 & 0 & 7 \end{pmatrix}$,

7) $\begin{pmatrix} -1 & -1 & 5 & -2 \\ 0 & 1 & 2 & 3 \\ 4 & 8 & -12 & 20 \\ 3 & 7 & 7 & 18 \\ 1 & 3 & -1 & 8 \end{pmatrix}$, 8) $\begin{pmatrix} 7 & 2 & -4 & 5 \\ 3 & 1 & 2 & 3 \\ 0 & 0 & 1 & 7 \\ 8 & -12 & 1 & 4 \\ 3 & -1 & 0 & 1 \end{pmatrix}$, 9) $\begin{pmatrix} -1 & -2 & 3 & -5 \\ 1 & 3 & -1 & 8 \\ 7 & 2 & -3 & 12 \\ -2 & -4 & 6 & -10 \\ 7 & 2 & -4 & 5 \end{pmatrix}$,

10) $\begin{pmatrix} -1 & -1 & 5 & -2 \\ 0 & 1 & 2 & 3 \\ 4 & 8 & -12 & 20 \\ 3 & 7 & 7 & 18 \\ 1 & 3 & -1 & 8 \end{pmatrix}$, 11) $\begin{pmatrix} -1 & -1 & -5 & 5 \\ 3 & 1 & 2 & 3 \\ 20 & 8 & -12 & 7 \\ 18 & 7 & 7 & 0 \\ 8 & 3 & -1 & 0 \end{pmatrix}$, 12) $\begin{pmatrix} 0 & 7 & -3 & -1 \\ 1 & 3 & 2 & 0 \\ 8 & -12 & 1 & 4 \\ 7 & 7 & 0 & 3 \\ 3 & -1 & 0 & 1 \end{pmatrix}$.

6. Berilgan matritsaning rangini elementar almashtirishlar yordamida toping

1) $\begin{pmatrix} 1 & 2 & -3 & 5 \\ 3 & 7 & -7 & 18 \\ -2 & -2 & 7 & -12 \\ 7 & 14 & -21 & 35 \end{pmatrix}$, 2) $\begin{pmatrix} 1 & 5 & 3 & 11 \\ 0 & 1 & 2 & 3 \\ 0 & 9 & 19 & 25 \\ 0 & 0 & -2 & 4 \end{pmatrix}$, 3) $\begin{pmatrix} 1 & 2 & -3 & 5 \\ -1 & -1 & 5 & -2 \\ 0 & -1 & -1 & -1 \\ 3 & 3 & -15 & 6 \end{pmatrix}$,

4) $\begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & 4 & 8 & 12 \\ -2 & -4 & 6 & -10 \\ 7 & 21 & -7 & 56 \end{pmatrix}$, 5) $\begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 7 \end{pmatrix}$, 6) $\begin{pmatrix} 3 & 2 & -2 & 3 \\ 1 & 1 & 2 & 3 \\ 3 & 2 & -1 & 1 \\ 1 & 1 & 0 & 7 \end{pmatrix}$,

7) $\begin{pmatrix} -1 & -1 & 5 & -2 \\ 0 & 1 & 2 & 3 \\ 4 & 8 & -12 & 20 \\ 3 & 7 & 7 & 18 \\ 1 & 3 & -1 & 8 \end{pmatrix}$, 8) $\begin{pmatrix} 7 & 2 & -4 & 5 \\ 3 & 1 & 2 & 3 \\ 0 & 0 & 1 & 7 \\ 8 & -12 & 1 & 4 \\ 3 & -1 & 0 & 1 \end{pmatrix}$, 9) $\begin{pmatrix} -1 & -2 & 3 & -5 \\ 1 & 3 & -1 & 8 \\ 7 & 2 & -3 & 12 \\ -2 & -4 & 6 & -10 \\ 7 & 2 & -4 & 5 \end{pmatrix}$,

10) $\begin{pmatrix} -1 & -1 & 5 & -2 \\ 0 & 1 & 2 & 3 \\ 4 & 8 & -12 & 20 \\ 3 & 7 & 7 & 18 \\ 1 & 3 & -1 & 8 \end{pmatrix}$, 11) $\begin{pmatrix} -1 & -1 & -5 & 5 \\ 3 & 1 & 2 & 3 \\ 20 & 8 & -12 & 7 \\ 18 & 7 & 7 & 0 \\ 8 & 3 & -1 & 0 \end{pmatrix}$, 12) $\begin{pmatrix} 0 & 7 & -3 & -1 \\ 1 & 3 & 2 & 0 \\ 8 & -12 & 1 & 4 \\ 7 & 7 & 0 & 3 \\ 3 & -1 & 0 & 1 \end{pmatrix}$.

O`z-o`zini tekshirish uchun savollar

1. Matritsaning qanday turlarini bilasiz?
2. Matritsalar ustida qanday amallar bajarish mumkin?
3. Matritsalar ustida qanday elementar almashtirishlar bajariladi?
4. Transponirlangan matritsa deb qanday matrisaga aytiladi?
5. O‘rin almashtirishlar va ulardagi inversiyalar soni qanday aniqlanadi?
6. Kvadrat matritsaning determinanti deb nima aytiladi?
7. Teskari matritsa qanday topiladi?

O`z-o`zini tekshirish uchun testlar

Matrisalarga oid testlar

1. Agar $A = \begin{pmatrix} -2 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$, bo‘lsa, $A \cdot B =?$

a) $\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 2 & -5 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}$ c) $\begin{pmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -5 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

2. Matrisaning mos satrlarini mos ustunlari bilan almashtirilsa … matrisa hosil bo‘ladi.
a) transponirlangan b) teskari c) birlik d) simmetrik
3. Agar berilgan matrisa o‘zining transponirlangan matrisasiga teng bo‘lsa, bunday matrisa... matritsa deyiladi.
a) nol b) teskari c) birlik d) simmetrik
4. Matritsani shu matritsaning teskarisiga ko‘paytirilsa ... matritsa hosil bo‘ladi
a) nol b) transponirlangan c) birlik d) simmetrik
5. Qanday matritsaning teskari matritsasi o‘ziga teng bo‘ladi?
a) nol b) kvadrat c) birlik d) simmetrik
6. Quyidagi matritsalarning qaysilarining teskarisi mavjud emas: 1) ustun-matrisa, 2) birlik matrisa, 3) nol matrisa, 4) xosmas matrisa, 5) satr-matrisa
a) 1),3)va 5) b) 2) va4) c) 1) va 3) d) 1) va3)
7. (2x3) o‘lchamli matritsani (3x2) o‘lchamli matritsaga ko‘paytirish natijasida hosil bo‘ladigan matritsaning o‘lchamini aniqlang.
a) (3x3) b) (2x2) c) (3x2) d) (2x3)

8. (3x3) o‘lchamli matritsani (3x4) o‘lchamli matritsaga ko‘paytirish natijasida hosil bo‘ladigan matritsaning o‘lchamini aniqlang.

a) (3x3) b) (2x3) c) (3x4) d) (4x3)

9. Berilgan matritsaning noldan farqli minorlarining yuqori tartibiga uning...deyiladi.

a) rangi b) tartibi c) noli d) to‘ldiruvchisi

10. Quyidagi matritsalarning qaysilarining teskari matritsasini topish mumkin. 1) ustun, 2) birlik, 3) nol, 4) xosmas, 5) satr

a) 1) va 3) b) 1) va5) c) 2) va 4) d) 3) va 5)

11. $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ matritsada nechta satr bor?

a) 3 ta b) n ta c) m ta d) 4 ta

12. $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ matritsada nechta ustun bor?

a) 3 ta b) n ta c) m ta d) 4 ta

13. $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ matritsaning o‘lchamini toping.

a) $m \times n$ b) $n \times m$ c) $k \times n$ a) $m \times k$

14. $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ matritsaning birinchi satri elementlarni ko‘rsating.

a) $a_{11} \quad a_{12} \quad \dots \quad a_{1n}$ b) $a_{21} \quad a_{22} \quad \dots \quad a_{2n}$

c) $a_{m1} \quad a_{m2} \quad \dots \quad a_{mn}$ d) $\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}$

15. $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ matritsaning m-satri satri elementlarni ko‘rsating.

a) $a_{11} \quad a_{12} \quad \dots \quad a_{1n}$ b) $a_{21} \quad a_{22} \quad \dots \quad a_{2n}$

c) $a_{m1} \quad a_{m2} \quad \dots \quad a_{mn}$ d) $\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}$

16. $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ matritsaning 2-satri elementlarni ko‘rsating.

a) $a_{11} \quad a_{12} \quad \dots \quad a_{1n}$ b) $a_{21} \quad a_{22} \quad \dots \quad a_{2n}$

c) $a_{m1} \quad a_{m2} \quad \dots \quad a_{mn}$ d) $\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}$

17. $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ matritsaning 1-ustuni elementlarni ko‘rsating.

a) $a_{11} \quad a_{12} \quad \dots \quad a_{1n}$ b) $a_{21} \quad a_{22} \quad \dots \quad a_{2n}$

c) $a_{m1} \quad a_{m2} \quad \dots \quad a_{mn}$ d) $\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}$

18. Agar matritsada satrlar soni ustunlar soni teng bo‘lsa, u holda bunday matritsalar … deb yuritiladi.

a) nol b) kvadrat c) birlik d) simmetrik

19. Kvadrat matritsaning bosh diagonali faqat 1 lardan tashkil topgan bo‘lib, matritsaning qolgan elementlari 0 lardan iborat bo‘lsa, bu kvadrat matritsa ... matritsa deyiladi.

a) nol b) kvadrat c) birlik d) simmetrik

20. Matrisalar orasida ko‘paytirish amali qaysi hollarda aniqlangan?

a) Birinchi ko‘paytuvchi ustunlar soni ikkinchi ko‘paytuvchi satrlar soniga teng bo‘lganda

b) Birinchi ko‘paytuvchi satrlar soni ikkinchi ko‘paytuvchi ustunlar soniga teng bo‘lganda

c) Faqatgina kvadrat matrisalar qaralganda

d) Matrisalar orasida ko‘paytirish amali aniqlanmagan

21. Kvadrat matrisalar orasida ko‘paytirish amali qaysi hollarda aniqlangan?

a) Ularning tartiblari teng bo‘lganda

b) Birinchi ko‘paytuvchi tartibi ikkinchinikidan katta bo‘lganda

c) Birinchi ko‘paytuvchi tartibi ikkinchinikidan kichik bo‘lganda

d) Kvadrat matrisalar orasida ko‘paytirish amali aniqlanmagan

22. Matrisani skalyarga ko‘paytirish deganda … tushuniladi.

a) uning har bir elementini shu songa ko‘paytirish

b) uning birinchi satr elementlarini shu songa ko‘paytirish

c) uning birinchi ustun elementlarini shu songa ko‘paytirish

d) uning birinchi satridagi birinchi elementini shu songa ko‘paytirish

23. Agar $A = \begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix}$, $B = \begin{pmatrix} -3 & -9 \\ -2 & -4 \end{pmatrix}$ bo‘lsa, u holda $2A + 3B$ ni toping.

a) $\begin{pmatrix} -5 & -19 \\ 0 & 6 \end{pmatrix}$ b) $\begin{pmatrix} 0 & -1 \\ 1 & 5 \end{pmatrix}$ c) $\begin{pmatrix} 5 & -6 \\ 1 & 19 \end{pmatrix}$ d) $\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$

24. Agar $A = \begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix}$, $B = \begin{pmatrix} -3 & -9 \\ -2 & -4 \end{pmatrix}$ bo‘lsa, u holda $3A - 2B$ ni toping.

a) $\begin{pmatrix} 12 & 30 \\ 13 & 35 \end{pmatrix}$ b) $\begin{pmatrix} 10 & -12 \\ 16 & 35 \end{pmatrix}$ c) $\begin{pmatrix} 25 & -16 \\ 21 & 19 \end{pmatrix}$ d) $\begin{pmatrix} 12 & 13 \\ 15 & 16 \end{pmatrix}$

25. Agar $A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$ bo‘lsa, u holda $A - \lambda E$ ni toping.

a) $\begin{pmatrix} 2-\lambda & -1 & 2 \\ 5 & -3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{pmatrix}$ b) $\begin{pmatrix} 2+\lambda & -1 & 2 \\ 5 & -3+\lambda & 3 \\ -1 & 0 & -2+\lambda \end{pmatrix}$

c) $\begin{pmatrix} 2-\lambda & -1-\lambda & 2-\lambda \\ 5-\lambda & -3-\lambda & 3-\lambda \\ -1-\lambda & 0-\lambda & -2-\lambda \end{pmatrix}$ d) $\begin{pmatrix} 2+\lambda & -1-\lambda & 2-\lambda \\ 5-\lambda & -3+\lambda & 3-\lambda \\ -1-\lambda & 0-\lambda & -2+\lambda \end{pmatrix}$

26. Agar $A = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$ bo‘lsa, u holda AB ni toping.

a) $\begin{pmatrix} 5 & 2 \\ 7 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 29 & -22 \\ 31 & -24 \end{pmatrix}$ c) $\begin{pmatrix} 25 & -16 \\ 21 & 19 \end{pmatrix}$ d) $\begin{pmatrix} 12 & 13 \\ 15 & 16 \end{pmatrix}$

27. Agar $A = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$ bo‘lsa, u holda BA ni toping.

a) $\begin{pmatrix} 5 & 2 \\ 7 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 29 & -22 \\ 31 & -24 \end{pmatrix}$ c) $\begin{pmatrix} 25 & -16 \\ 21 & 19 \end{pmatrix}$ d) $\begin{pmatrix} 12 & 13 \\ 15 & 16 \end{pmatrix}$

28. Agar $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ -2 & 2 \\ -1 & 0 \end{pmatrix}$ bo‘lsa, u holda AB ni toping.

a) ko‘paytrib bo‘lmaydi b) $\begin{pmatrix} 10 & 14 \\ 4 & 4 \\ -1 & -2 \end{pmatrix}$ c) $\begin{pmatrix} 25 & -16 \\ 21 & 19 \end{pmatrix}$ d) $\begin{pmatrix} 12 & 13 \\ 15 & 16 \end{pmatrix}$

29. Agar $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ -2 & 2 \\ -1 & 0 \end{pmatrix}$ bo‘lsa, u holda BA ni toping.

a) ko‘paytrib bo‘lmaydi b) $\begin{pmatrix} 10 & 14 \\ 4 & 4 \\ -1 & -2 \end{pmatrix}$ c) $\begin{pmatrix} 25 & -16 \\ 21 & 19 \end{pmatrix}$ d) $\begin{pmatrix} 12 & 13 \\ 15 & 16 \end{pmatrix}$

30. Agar $A = (1 \quad 2 \quad 3)$, $B = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ bo‘lsa, u holda BA ni toping.

a) ko‘paytrib bo‘lmaydi b) (10) c) $\begin{pmatrix} 0 & 6 \\ 1 & 10 \end{pmatrix}$ d) (2)

3-§. CHIZIQLI ALGEBRAIK TENGLAMALAR SISTEMASI VA UNI TEKSHIRISH

Chiziqli algebraik tenglamalar sistemalariga oid masalalar I

1-masala. Ma'lumki, Covid-19 nomi bilan "mashhur" bo'lgan tojdor virus butun dunyo pandemiyasini keltirib chiqardi. O'zbekistonda ham 2020 yilning 15 martida tojdor virus aniqlangan birinchi holat qayd etilishi bilanoq unga oid bir qator qat'iy choralar belgilandi. Jumladan, 2020 yil 16 martdan barcha davlat, nodavlat va xususiy ta'lim muassasalarida, jumladan bog'cha, maktab, kollej, litsey va OTMlarda muddatdan oldin ta'til e'lon qilindi va o'qishlar to'xtatildi; 2020 yil 23 martdan boshlab barcha korxona va muassasalarga asosiy ish faoliyatiga ta'sir etmagan holda ishchi va xodimlarni qonunchilikda belgilangan tartibda mehnat ta'tiliga chiqarildi, qolgan ishchilarning faoliyatini maksimal darajada masofadan turib (onlayn yoki distansion tarzda) tashkil etildi.

Shunday sharoitda 2020 yil 1 may kuni Sardoba suv omborida to'g'on yorilishi natijasida halokat ro'y berdi. Jabr ko'rganlar uchun ko'p qavatli uylar quriladigan bo'ldi. Dastlab qurilayotgan uylarda ishlayotgan beton quyuvchi va chilangar (svarkachi)lar jami soni 450 nafar edi. Karantin sharoitlari yumshagani hamda aholini tezroq uy-joy bilan ta'minlash maqsadida beton quyuvchilar soni 2 barobar, chilangarlar soni esa 5 barobar oshirildi va ularning umumiy soni 1170 nafar bo'ldi. Qurilish ishlariga necha nafar beton quyuvchilar va necha nafar chilangarlar jalb etilganligini aniqlang.

Echish. Dastlabki:

beton quyuvchilar soni x;
chilangarlar soni y

bo'lsin. Keyinchalik esa:

beton quyuvchilar soni $2x$;
chilangarlar soni $5y$

bo'lgan. Masala shartlariga ko'ra:

dastlab $x + y = 450$,
keyin $2x + 5y = 1170$

munosabatlarga egamiz.

Masalani yechish ushbu

$$\begin{cases} x + y = 450, \\ 2x + 5y = 1170 \end{cases}$$

chiziqli tenglamalar sistemasini yechishga olib keladi.

Bu tenglamalar sistemasini yechishning bir necha usulini keltiramiz.

1-usul (Gauss usuli). Bu tenglamalar sistemasining asosiy matritsasi

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix},$$

kengaytirilgan matritsasi

$$A_{keng} = \begin{pmatrix} 1 & 1 & 450 \\ 2 & 5 & 1170 \end{pmatrix}.$$

A matritsaning rangi 2 ga teng, chunki undagi satrlar $((1 \quad 1)$ va $(2 \quad 5))$ hamda ustunlar $\left(\begin{pmatrix}1\\2\end{pmatrix} va \begin{pmatrix}1\\5\end{pmatrix}\right)$ chiziqli erkli. YA'ni $rank\, A = rank\, A_{keng} = 2$. Shuning uchun Kroneker-Kapelli teoremasiga ko'ra bu sistema yagona yechimga ega. Shu yechimni topish uchun kengaytirilgan matritsa ustida elementar almashtirishlar bajaramiz:

$$\begin{pmatrix}1 & 1 & 450\\2 & 5 & 1170\end{pmatrix}\Big|\begin{matrix}\times 2\\-\end{matrix} \sim \begin{pmatrix}1 & 1 & 450\\0 & 3 & 270\end{pmatrix}.$$

Demak, berilgan sistema ushbu

$$\begin{cases} x + y = 450, \\ \quad 3y = 270 \end{cases}$$

sistemaga teng kuchli (Gauss usulining asosi shunda). Oxirgi sistemaning ikkinchi tenglamasidan $y = 90$ bo'lib, y ning bu qiymati sistemadagi birinchi tenglamaga qo'yilsa, $x = 360$ bo'ladi.

Javob: Qurilish ishlarida

Dastlab 360 nafar beton quyuvchi, 90 nafar chilangar,

keyin $2 \times x = 720$ nafar beton quyuvchi, $5 \times y = 450$ nafar chilangar ishlagan.

2-usul (Kramer usuli). Hosil qilingan chiziqli tenglamalar sistemasida tenglamalar soni (2 ta) bilan o'zgaruvchilar soni (2 ta) teng bo'lgani uchun determinatlar usulida foydalanish mumkin.

$$\Delta = \begin{vmatrix}1 & 1\\2 & 5\end{vmatrix} = 5 - 2 = 3,$$

$$\Delta_1 = \Delta_x = \begin{vmatrix}450 & 1\\1170 & 5\end{vmatrix} = 2250 - 1170 = 1080,$$

$$\Delta_2 = \Delta_y = \begin{vmatrix}1 & 450\\2 & 1170\end{vmatrix} = 1170 - 900 = 270.$$

Bundan Kramer formulalariga ko'ra

$$x = \frac{\Delta_x}{\Delta} = \frac{1080}{3} = 360,$$

$$y = \frac{\Delta_y}{\Delta} = \frac{270}{3} = 90.$$

Javob: Qurilish ishlarida

Dastlab 360 nafar beton quyuvchi, 90 nafar chilangar,

keyin 720 nafar beton quyuvchi, 450 nafar chilangar ishlagan.

3-usul (Matritsalar usuli). Masala shartlaridan hosil qilingan tenglamalar sistemasi uchun ushbularni yozib olamiz:

$A = \begin{pmatrix}1 & 1\\2 & 5\end{pmatrix}$ – tenglamalar sistemasining asosiy matritsasi,

$X = \begin{pmatrix}x\\y\end{pmatrix}$ – tenglamalar sistemasining o'zgaruvchilari (ustun-)matritsasi,

$B = \begin{pmatrix}450\\1170\end{pmatrix}$ – tenglamalar sistemasining ozod hadlari (ustun-) matritsasi.

U holda qaralayotgan tenglamalar sistemasini

$$\begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 450 \\ 1170 \end{pmatrix}$$

matritsa ko‘rinishida. yoki qisqalik uchun

$$AX = B$$

ko‘rinishda yozish mumkin. Agar A matritsaning A^{-1} teskarisi mavjud bo‘lsa, $AX = B$ tenglamaning echimi

$$X = A^{-1} \cdot B$$

kabi topiladi (matritsalar usulining mag‘zi mana shunda).

A^{-1} mavjud bo‘lish sharti $\det A \neq 0$ hisoblanadi:

$$\det A = \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} = 5 - 2 = 3 \neq 0.$$

Teskari matritsaning o‘zi esa

$$A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$$

formuladan topiladi. Bu erda A_{11} ifoda a_{11} elementning algebraik to‘ldirmasi, A_{12} ifoda a_{12} elementning algebraik to‘ldirmasi, A_{21} ifoda a_{21} elementning algebraik to‘ldirmasi, A_{22} ifoda a_{22} elementning algebraik to‘ldirmasi. Qaralayotgan misolda,

$$A_{11} = a_{22} = 5, \quad A_{12} = -a_{21} = -2, \quad A_{21} = -a_{12} = -1, \quad A_{22} = a_{11} = 1$$

bo‘lgani uchun

$$A^{-1} = \frac{1}{3} \cdot \begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Demak,

$$X = \begin{pmatrix} \frac{5}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 450 \\ 1170 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \cdot 450 - \frac{1}{3} \cdot 1170 \\ -\frac{2}{3} \cdot 450 + \frac{1}{3} \cdot 1170 \end{pmatrix} = \begin{pmatrix} 750 - 390 \\ -300 + 390 \end{pmatrix} = \begin{pmatrix} 360 \\ 90 \end{pmatrix},$$

ya’ni $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 360 \\ 90 \end{pmatrix}$ bo‘lib, bu erdan $x = 360$ va $y = 90$ kelib chiqadi.

Javob: Qurilish ishlarida

Dastlab 360 nafar beton quyuvchi, 90 nafar chilangar,

keyinchalik 720 nafar beton quyuvchi, 450 nafar chilangar ishlagan.

2-masala. Dizayner bitta xonaning ichki dizayni uchun 2 kun, bitta uyning tashqi dizayni uchun 3 kun, bitta hovli landshaft dizayni uchun esa 5 kun sarflaydi. Agar xonalarning ichki dizayni bo‘yicha buyurtmalar uyning tashqi dizayni bo‘yicha buyurtmalardan 3 taga ko‘p, hovli landshaft dizayni bo‘yicha buyurtmalardan 4 taga ortiq bo‘lib, jami buyurtmalarni bajarishga 21 kun sarflangan bo‘lsa, dizayner nechta xonaning ichki dizayni, nechta uyning tashqi dizayni va nechta hovlining landshaft dizayni bo‘yicha buyurtma bajargan?

Yechish.

ichki dizayn qilinadigan xonalar soni x,

tashqi dizayn qilinadigan uylar soni y,

landshaft dizayni qilinadigan hovlilar soni z

bo‘lsin. Masala shartlariga ko‘ra

$$x = y + 3,$$
$$x = z + 4,$$
$$2x + 3y + 5z = 21$$

bo‘ladi. Demak, masala ushbu tenglamalar sistemasini echishga olib kelinadi:

$$\begin{cases} x - y = 3, \\ x - z = 4, \\ 2x + 3y + 5z = 21. \end{cases}$$

Hosil qilingan bu sistemaning asosiy matritsasi

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & 5 \end{pmatrix},$$

kengaytirilgan matritsasi esa

$$A_{keng} = \begin{pmatrix} 1 & -1 & 0 & 3 \\ 1 & 0 & -1 & 4 \\ 2 & 3 & 5 & 21 \end{pmatrix}$$

bo‘ladi. Elementar almashtirishlardan so‘ng

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 5 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix},$$

A matritsaning rangi 3 ga tengligini ko‘ramiz: $rang\ A = 3$. U holda kengaytirilgan matritsaning rangi ham 3 ga teng bo‘ladi: $rang\ A_{keng} = 3$. Kroneker-Kapelli teoremasidan bu sistemaning yechimga egaligi kelib chiqadi.

1-usul (Gauss usuli).

$$A_{keng} = \begin{pmatrix} 1 & -1 & 0 & 3 \\ 1 & 0 & -1 & 4 \\ 2 & 3 & 5 & 21 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 5 & 5 & 15 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix} \sim$$
$$\sim \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Bundan

$$\begin{cases} x - y = 3, \\ x - z = 4, \\ 2x + 3y + 5z = 21 \end{cases} \Leftrightarrow \begin{cases} x - y = 3, \\ y - z = 1, \\ z = 1 \end{cases} \Rightarrow \begin{cases} x = 5 \\ y = 2 \\ z = 1 \end{cases}$$

Javob: Buyurtmalar quyidagicha bo‘lgan: 5 ta xonaning ichki dizayni, 2 ta uyning tashqi dizayni, 1 ta hovlining landshaft dizayni qilingan.

2-usul (Kramer usuli).

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & 5 \end{vmatrix} = 0 + 2 + 0 - 0 + 5 + 3 = 10,$$

$$\Delta_x = \begin{vmatrix} 3 & -1 & 0 \\ 4 & 0 & -1 \\ 21 & 3 & 5 \end{vmatrix} = 0 + 21 + 0 - 0 + 20 + 9 = 50,$$

$$\Delta_y = \begin{vmatrix} 1 & 3 & 0 \\ 1 & 4 & -1 \\ 2 & 21 & 5 \end{vmatrix} = 20 - 6 + 0 - 0 - 15 + 21 = 20,$$

$$\Delta_z = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 0 & 4 \\ 2 & 3 & 21 \end{vmatrix} = 0 - 8 + 9 - 0 - 12 + 21 = 10.$$

$$x = \frac{\Delta_x}{\Delta} = 1, \quad y = \frac{\Delta_y}{\Delta} = 5, \quad z = \frac{\Delta_z}{\Delta} = 1.$$

Javob: $x = 1,\ y = 5,\ z = 1,$ ya'ni buyurtmalar quyidagicha bo'lgan: 5 ta xonaning ichki dizayni, 2 ta uyning tashqi dizayni, 1 ta hovlining landshaft dizayni qilingan.

3-usul (Matritsalar usuli). Masala shartlaridan hosil qilingan tenglamalar sistemasi uchun ushbularni yozib olamiz:

$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & 5 \end{pmatrix}$ – tenglamalar sistemasining asosiy matritsasi,

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ – tenglamalar sistemasining o'zgaruvchilari (ustun-)matritsasi,

$B = \begin{pmatrix} 3 \\ 4 \\ 21 \end{pmatrix}$ – tenglamalar sistemasining ozod hadlari (ustun-) matritsasi.

U holda qaralayotgan tenglamalar sistemasini

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

matritsa ko'rinishida. yoki qisqalik uchun

$$AX = B$$

ko'rinishda yozish mumkin. Agar A matritsaning A^{-1} teskarisi mavjud bo'lsa, $AX = B$ tenglamaning echimi

$$X = A^{-1} \cdot B$$

kabi topiladi (matritsalar usulining mag'zi mana shunda).

A^{-1} mavjud bo'lish sharti $\det A \neq 0$ hisoblanadi:

$$\det A = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & 5 \end{vmatrix} = 0 + 2 + 0 - 0 + 5 + 3 = 10 \neq 0.$$

Teskari matritsaning o'zi esa

$$A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

formuladan topiladi. Bu erda A_{ij} ifoda a_{ij} elementning algebraik to'ldirmasi, $i,\ j = 1,\ 2,\ 3$. Mazkur algebraik to'ldirmalarni hisoblaymiz:

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 3 & 5 \end{vmatrix} = 3,$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} = -7,$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3,$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} -1 & 0 \\ 3 & 5 \end{vmatrix} = 5,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} = 5,$$
$$A_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = -5,$$
$$A_{31} = (-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1,$$
$$A_{32} = (-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1,$$
$$A_{33} = (-1)^{3+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1.$$

Topilgan bu qiymatlarni formulaga qo‘yib, ushbu matritsani hosil qilamiz

$$A^{-1} = \frac{1}{10}\cdot\begin{pmatrix} 3 & 5 & 1 \\ -7 & 5 & 1 \\ 3 & -5 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & \frac{1}{2} & \frac{1}{10} \\ -\frac{7}{10} & \frac{1}{2} & \frac{1}{10} \\ \frac{3}{10} & -\frac{1}{2} & \frac{1}{10} \end{pmatrix}.$$

Hisoblab topilgan bu A^{-1} matritsa A matritsaga teskari bo‘ladi (Haqiqatan ham,

$$A\cdot A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & 5 \end{pmatrix}\cdot\begin{pmatrix} \frac{3}{10} & \frac{1}{2} & \frac{1}{10} \\ -\frac{7}{10} & \frac{1}{2} & \frac{1}{10} \\ \frac{3}{10} & -\frac{1}{2} & \frac{1}{10} \end{pmatrix} =$$

$$= \begin{pmatrix} 1\cdot\frac{3}{10} + (-1)\cdot\left(-\frac{7}{10}\right) + 0\cdot\frac{3}{10} & 1\cdot\frac{1}{2} + (-1)\cdot\frac{1}{2} + 0\cdot\left(-\frac{1}{2}\right) & 1\cdot\frac{1}{10} + (-1)\cdot\frac{1}{10} + 0\cdot\frac{1}{10} \\ 1\cdot\frac{3}{10} + 0\cdot\left(-\frac{7}{10}\right) + (-1)\cdot\frac{3}{10} & 1\cdot\frac{1}{2} + 0\cdot\frac{1}{2} + (-1)\cdot\left(-\frac{1}{2}\right) & 1\cdot\frac{1}{10} + 0\cdot\frac{1}{10} + (-1)\cdot\frac{1}{10} \\ 2\cdot\frac{3}{10} + 3\cdot\left(-\frac{7}{10}\right) + 5\cdot\frac{3}{10} & 2\cdot\frac{1}{2} + 3\cdot\frac{1}{2} + 5\cdot\left(-\frac{1}{2}\right) & 2\cdot\frac{1}{10} + 3\cdot\frac{1}{10} + 5\cdot\frac{1}{10} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E,$$

ya’ni $A\cdot A^{-1} = E$. Xuddi shuningdek, $A^{-1}\cdot A = E$ ekanligini ham ko‘rsatish mumkin). Demak,

$$X = A^{-1}\cdot B = \begin{pmatrix} \frac{3}{10} & \frac{1}{2} & \frac{1}{10} \\ -\frac{7}{10} & \frac{1}{2} & \frac{1}{10} \\ \frac{3}{10} & -\frac{1}{2} & \frac{1}{10} \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 4 \\ 21 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}\cdot 3 + \frac{1}{2}\cdot 4 + \frac{1}{10}\cdot 21 \\ -\frac{7}{10}\cdot 3 + \frac{1}{2}\cdot 4 + \frac{1}{10}\cdot 21 \\ \frac{3}{10}\cdot 3 - \frac{1}{2}\cdot 4 + \frac{1}{10}\cdot 21 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix},$$

ya’ni $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ bo‘lib, bu yerdan $x = 5$, $y = 2$ va $z = 1$ kelib chiqadi.

Javob: $x = 1$, $y = 5$, $z = 1$, ya’ni buyurtmalar quyidagicha bo‘lgan: 5 ta xonaning ichki dizayni, 2 ta uyning tashqi dizayni, 1 ta hovlining landshaft dizayni qilingan.

Chiziqli algebraik tenglamalar sistemasi va uni tekshirish (nazariy qism)

3.1. Chiziqli algebraik tenglamalar sistemasi

n ta $x_1, x_2, \ldots, x_n$ noma'lumlarning m ta tenglamasidan tuzilgan ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \ldots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \quad (1)$$

sistema **chiziqli algebraik tenglamalar sistemasi** deyiladi. Chiziqli algebraik tenglamalar sistemasini qisqalik uchun *tenglamalar sistemasi* yoki *sistema* deb ishlatamiz.

a_{ij} sonlar sistemaning **koeffitsientlari**, b_j sonlar esa sistemaning **ozod hadlari** deyiladi, $i = 1,\ 2,\ \ldots,\ n,\ j = 1,\ 2, \ldots, m$.

Agar barcha ozod hadlar nolga teng, ya'ni $\forall j \in \{1,\ 2, \ldots, m\}$ uchun $b_j = 0$ bo'lsa, sistema **bir jinsli** deyiladi. Agar kamida bitta ozod had noldan farqli bo'lsa, ya'ni $\exists j \in \{1,\ 2, \ldots, m\}$ uchun $b_j = 0$ bo'lsa, sistema **bir jinslimas** deyiladi ($\forall$ belgisi ixtiyoriylik kvantori deb ataladi va "ixtiyoriy", "barcha", "har qanday", "har bir" deb o'qiladi; $\exists$ belgisi mavjudlik kvantori deb ataladi va "shunday", "shunday ... mavjudki", "shunday ... topilib" kabi talqin qilinadi).

Tenglamalar sistemasining **yechimi** deb, $x_1,\ x_2,\ \ldots,\ x_n$ noma'lumlarning o'rniga qo'yganda sistemaning har bir tenglamasini ayniyatga aylantiruvchi tartiblangan $(\alpha_1,\ \alpha_2, \ldots,\ \alpha_n)$ sonlar jamlanmasiga aytiladi.

Ixtiyoriy bir jinsli tenglamalar sistemasi birgalikda bo'ladi. U hech bo'lmaganda (trivial yechim deb ataluvchi) $x_1 = x_2 = \cdots = x_n = 0$ yechimga ega.

Hech bo'lmaganda bitta yechimga ega sistema **birgalikda bo'lgan** tenglamalar sistemasi deyiladi. Bironta ham yechimga ega bo'lmagan sistema **birgalikda bo'lmagan** tenglamalar sistemasi deyiladi. Yagona yechimga ega sistema **aniq** sistema, cheksiz ko'p yechimga ega sistema **aniqmas** sistema deb yuritiladi.

3.2. Chiziqli algebraik tenglamalar sistemasi yozilishining matritsali shakli

(1) sistemaning koeffitsientlari, ozod hadlari va noma'lumlari ishtirokidagi quyidagi matritsalarni

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad A_{keng} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

kiritamiz.

A matritsa (1) sistemaning **asosiy** matritsasi deyiladi. Asosiy matritsa (1) sistemaning koeffitsientlarini ifodalaydi. A_{keng} matritsa (1) sistemaning

kengaytirilgan matritsasi deyiladi. Ravshanki, kengaytirilgan matritsa asosiy matritsaga ozod hadlar ustunini qo‘shishdan hosil qilinadi.

B ustun-matritsa (1) sistemaning ozod hadlari matritsasi, X ustun-matritsa (1) sistemaning noma’lumlari matritsasi deyiladi.

Endi (1) sistemani

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{2}$$

ko‘rinishda yozish mumkin. (2) tenglama (1) chiziqli algebraik tenglamalar sistemasining yozilishining **matritsali shakli** deyiladi.

Qisqalik uchun (2) tenglamani

$$A \cdot X = B \tag{2'}$$

ko‘rinishda yozish mumkin.

Tenglamalar sistemasi yechimga ega yoki ega emasligini oldindan aytish mushkul masala. Tenglamalar sistemasi birgalikda bo‘lishi yoki bo‘lmasligini aniqlash, agarda birgalikda bo‘lsa, yechimlari sonini ko‘rsatish jarayoni **tenglamalar sistemasini tekshirish** deb ataladi.

3.3. Chiziqli algebraik tenglamalar sistemasini tekshirish

Tenglamalar sistemasi birgalikda bo‘lishining alomatini **Kroneker-Kapelli teoremasi deb ataluvchi** quyidagi **teorema ifodalaydi.**

Teorema. Chiziqli algebraik tenglamalar sistemasi birgalikda bo‘lishi uchun uning asosiy va kengaytirilgan matritsalarining ranglari teng bo‘lishi zarur va etarli.

Agar (1) sistemani qarasak, bu teoremani quyidagicha bayon etish mumkin:

((1) sistema echimga ega) $\Leftrightarrow$ ($rang\ A = rang\ A_{keng}$).

(1) sistemada n ta noma’lum qatnashayotganligini e’tiborga olib, quyidagi **muhim** xulosalarni keltiramiz:

1. Agar $rang\ A \neq rang\ A_{keng}$ bo‘lsa, (1) sistema birgalikda bo‘lmaydi;
2. Agar $rang\ A = rang\ A_{keng} < n$ bo‘lsa, (1) sistema birgalikda, lekin aniqmas (ya’ni cheksiz ko‘p echimga ega) bo‘ladi;
3. Agar $rang\ A = rang\ A_{keng} = n$ bo‘lsa, (1) birgalikda bo‘lgan aniq sistema bo‘ladi (yagona echimga ega).

Sistema birgalikda bo‘lsa, uning yechimini topish masalasi paydo bo‘ladi.

3.4. Chiziqli algebraik tenglamalar sistemasini yechish

Chiziqli algebraik tenglamalar sistemasini echishning bir necha usullari mavjud. Umumiy o‘rta ta’limda sistemani yechishning o‘rniga qo‘yish usuli o‘rganilgan. Nostandart usulda yechish eng maqbul bo‘ladigan sistemalar ham mavjud.

Bir jinslimas chiziqli algebraik tenglamalar sistemasini yechishning Kramer usuli, teskari matritsa usuli, Gauss usuli va Gauss-Jordan usuli haqida to‘xtalib o‘tamiz.

3.4.1. Chiziqli algebraik tenglamalar sistemasini yechishning Kramer usuli

Chiziqli algebraik tenglamalar sistemasini yechishning Kramer usuli qo‘llash uchun sistemasidagi tenglamalar soni bilan noma’lumlar soni teng hamda asosiy matritsaning determinanti noldan farqli bo‘lishi talab etiladi.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \tag{3}$$

sistema uchun

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{vmatrix}, \quad \dots, \quad \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}$$

deteminatlar hisoblanadi va noma’lumlar Kramer formulalari deb ataluvchi ushbu

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \ \dots, \ x_n = \frac{\Delta_n}{\Delta}$$

tengliklar orqali topiladi. Talabga ko‘ra $\Delta \neq 0$ bo‘lgani uchun bu tengliklar ma’noga ega.

$\Delta = 0$ bo‘ladigan sistemalar uchun Kramer usulini qo‘llab bo‘lmaydi.

3.4.2. Chiziqli algebraik tenglamalar sistemasini yechishning teskari matritsa usuli

Chiziqli algebraik tenglamalar sistemasini yechishning teskari matritsa usuli qo‘llash uchun sistemasidagi tenglamalar soni bilan noma’lumlar soni teng hamda asosiy matritsaning determinanti noldan farqli bo‘lishi talab etiladi. Aytaylik, (1) sistemada $m = n$ bo‘lsin. U holda

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \tag{3}$$

sistemaga ega bo‘lamiz. (3) ning asosiy A, noma’lumlar X va ozod hadlar B matritsalari

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

bo‘lsin. U holda (3) sistemani

$$A \cdot X = B \tag{2'}$$

matritsali ko‘rinishda yozish mumkin. $(2')$ matritsali tenglamaning yechimi

$$X = A^{-1} \cdot B \qquad (4)$$

kabi topiladi va bu (4) yechim berilgan (3) sistemaning echimi bo‘ladi. Shu o‘rinda A^{-1} matritsa A matritsaga teskari ekanligini va

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

tenglikdan topilishini eslatib o‘tamiz. Bu erda

$$A_{ij} = (-1)^{i+j} \cdot \det\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1(j-1)} & a_{1(j+1)} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(j-1)} & a_{2(j+1)} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(j-1)} & a_{n(j+1)} & \dots & a_{nn} \end{pmatrix} -$$

a_{ij} elementning algebraik to‘ldirmasi, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$.

Demak, (4) tenglikni

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \qquad (4')$$

ko‘rinishda yozish mumkin. Bayon etilganlar chiziqli algebraik tenglamalar sistemasini yechishning teskari matritsa usuli hisoblanadi.

Berilgan (1) sistemadagi tenglamalar soni m bilan noma’lumlar soni n teng bo‘lmasa, yoki teng bo‘lsayu asosiy matritsaning determinanti nolga teng ($\det A = 0$) bo‘lsa, chiziqli algebraik tenglamalar sistemasini yechishning teskari matritsa usulini qo‘llab bo‘lmaydi.

3.4.3. Chiziqli algebraik tenglamalar sistemasini yechishning Gauss usuli

Gauss usulida quyidagi elementar almashtirishlarga ruxsat etiladi:

($e'1$) Matritsaning faqatgina satrlarining o‘rinlarini almashtirish;

($e'2$) Matritsaning faqatgina biror satrini noldan farqli songa ko‘paytirish;

($e'3$) Matritsaning faqatgina biror satri elementlariga boshqa biror satri mos elementlarini biror tayinli songa ko‘paytirib qo‘shish;

($e'4$) Matritsada faqatgina nol-satrlar bo‘lsa, ularni o‘chirish.

Diqqat! Matritsaning ustunlarining o‘rinlarini almashtirishga; biror ustunini noldan farqli songa ko‘paytirishga; biror ustuni elementlariga boshqa biror ustuni mos elementlarini biror tayinli songa ko‘paytirib qo‘shishga; nol-ustunlarni o‘chirishga yo‘l qo‘yilmaydi.

Ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

sistemaning kengaytirilgan

$$A_{keng} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2(j-1)} & a_{2j} & \dots & a_{2n} & b_2 \\ \cdots & \cdots & \dots & \cdots & \cdots & \dots & \cdots & \cdots \\ a_{k1} & a_{k2} & \dots & a_{k(j-1)} & a_{k(j-1)} & \dots & a_{kn} & b_k \\ a_{(k+1)1} & a_{(k+1)2} & \dots & a_{(k+1)(j-1)} & a_{(k+1)j} & \dots & a_{(k+1)j} & b_{k+1} \\ \cdots & \cdots & \dots & \cdots & \cdots & \dots & \cdots & \cdots \\ a_{m1} & a_{m2} & \dots & a_{m(j-1)} & a_{mj} & \dots & a_{mn} & b_m \end{pmatrix}$$

matritsasini elementar almashtirishlar orqali

$$\begin{pmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1(j-1)}^* & a_{1j}^* & \cdots & a_{1n}^* & b_1^* \\ 0 & a_{22}^* & \cdots & a_{2(j-1)}^* & a_{2j}^* & \cdots & a_{2n}^* & b_2^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{kj}^* & \cdots & a_{kn}^* & b_k^* \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1(j-1)}^* & a_{1j}^* & \cdots & a_{1n}^* & b_1^* \\ 0 & a_{22}^* & \cdots & a_{2(j-1)}^* & a_{2j}^* & \cdots & a_{2n}^* & b_2^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{kj}^* & \cdots & a_{kn}^* & b_k^* \end{pmatrix} \tag{5}$$

pog‘onali shakl ko‘rinishiga olib kelish chiziqli algebraik tenglamalar sistemasini echishning Gauss usulining asosi hisoblanadi. Natijada (1) sistema o‘ziga teng kuchli bo‘lgan ushbu

$$\begin{cases} a_{11}^*x_1 + a_{12}^*x_2 + \cdots + a_{1(j-1)}^*x_{j-1} + a_{1j}^*x_j + \cdots + a_{1n}^*x_n = b_1^*, \\ a_{22}^*x_2 + \cdots + a_{2(j-1)}^*x_{j-1} + a_{2j}^*x_j + \cdots + a_{2n}^*x_n = b_2^* \\ \dots, \\ a_{ij}^*x_j + \cdots + a_{in}^*x_n = b_i^* \end{cases}$$

sistemani yechishga olib kelinadi.

Quyidagi hollar bo‘lishi mumkin:

1-hol. (5) matritsa ushbu ko‘rinishda bo‘lsin

$$\begin{pmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1(j-1)}^* & a_{1j}^* & \cdots & a_{1n}^* & b_1^* \\ 0 & a_{22}^* & \cdots & a_{2(j-1)}^* & a_{2j}^* & \cdots & a_{2n}^* & b_2^* \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & 0 & a_{(k-1)j}^* & \cdots & a_{(k-1)n}^* & b_{k-1}^* \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & b_k^* \end{pmatrix}.$$

Bu holda, ravshanki, $rang\ A = k - 1$ va $rang\ A_{keng} = k$ bo‘ladi (**tushuntiring!**). Bundan $rang\ A \neq rang\ A_{keng}$ ekanligi kelib chiqadi. Kroneker-Kapelli teoremasiga ko‘ra, sistema yechimga ega emas.

2-hol. (5) matritsa ushbu ko‘rinishda bo‘lsin

$$\begin{pmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1(j-1)}^* & a_{1j}^* & \cdots & a_{1(n-1)}^* & a_{1n}^* & b_1^* \\ 0 & a_{22}^* & \cdots & a_{2(j-1)}^* & a_{2j}^* & \cdots & a_{2(n-1)}^* & a_{2n}^* & b_2^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{kn}^* & b_k^* \end{pmatrix}, \qquad a_{kn}^* \neq 0.$$

Bu holda, ravshanki, $rang\ A = rang\ A_{keng} = k$ bo‘ladi (**tushuntiring!**). Bundan Kroneker-Kapelli teoremasiga ko‘ra, sistema yagona yechimga ega.

3-hol. (5) matritsa ushbu ko‘rinishda bo‘lsin

$$\begin{pmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1(j-1)}^* & a_{1j}^* & \cdots & a_{1n}^* & b_1^* \\ 0 & a_{22}^* & \cdots & a_{2(j-1)}^* & a_{2j}^* & \cdots & a_{2n}^* & b_2^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{kj}^* & \cdots & a_{kn}^* & b_k^* \end{pmatrix}.$$

Bu holda, ravshanki, $rang\ A = rang\ A_{keng} = k < n$ bo‘ladi (**tushuntiring!**). Bundan Kroneker-Kapelli teoremasiga ko‘ra, sistema yechimga cheksiz ko‘p yechimga ega bo‘ladi. Shuni ta’kidlash lozimki, x_1, x_2, …, x_k noma’lumlarga mos echimlar x_{k+1}, x_{k+2}, …, x_n noma’lumlarga bog‘liq bo‘ladi. Bu holat o‘ziga xos geometrik talqinga ega.

3.4.4. Chiziqli algebraik tenglamalar sistemasini yechishning Gauss-Jordan usuli

Chiziqli algebraik tenglamalar sistemasini yechishning Gauss-Jordan usuli Gauss usulining rivojlantirilganidir. Bu usulda ham $(e'1)$ – $(e'4)$ elementar almashtirishlar bajarishga to‘g‘ri keladi.

Bu usulda ham

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \qquad\qquad \dots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

sistemaning

$$A_{keng} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

kengaytirilgan matritsasi bilan ishlaymiz.

Ushbu tushunchalar talab etiladi. Nol bo‘lmagan satrning noldan farqli birinchi elementi shu satrning **yetakchi** elementi deyiladi. Nol bo‘lmagan ustunning noldan farqli ixtiyoriy elementi shu ustunning **yechuvchi** elementi deyiladi.

Birinchi qadamda birinchi ustunning yechuvchi elementi topiladi. Aytaylik, bu yechuvchi element a_{i1} bo‘lsin. Agar $a_{i1} \neq 1$ bo‘lsa, i-satrni a_{i1} ga bo‘lamiz. Keyin elementar almashtirishlar orqali birinchi ustundagi i-elementdan tashqari barcha elementlarini nolga aylantiramiz.

Ikkinchi qadamda ikkinchi ustunning yechuvchi elementi topiladi. Shuni ta’kidlash lozimki, bu element i-satrdan olinishi mumkin emas. Har bitta satrdan bittagina yechuvchi element olishga ruxsat etiladi. Agar ikkinchi ustunda i-satrdan boshqa hamma satrlardagi element nolga teng bo‘lsa, uchinchi ustunga o‘tamiz. Uchinchi ustunda ham shu shartlarni qanoatlantiruvchi etakchi element topilmasa, to‘rtinchi ustunga, va hakozo ustunga o‘tiladi.

Aytaylik, k-ustunda yuqoridagi talablarni qanoatlantiruvchi $a_{lk} \neq 0$ yechuvchi element mavjud bo‘lsin (shartlarga ko‘ra $k \neq 1$, $l \neq i$). Agar $a_{lk} \neq 1$ bo‘lsa, l-satrni a_{lk} ga bo‘lamiz. Keyin elementar almashtirishlar orqali k-ustundagi l-elementdan tashqari barcha elementlarini nolga aylantiramiz.

Bu jarayon uzog‘i bilan n qadamdan keyin tugaydi (**nega?**).

3.4.4.1.-misol. Ushbu sistemani Gauss-Jordan usuli yordamida yeching

$$\begin{cases} 3x_1 + 2x_2 + 8x_3 = 4, \\ -5x_1 + 7x_2 + 2x_3 = -43, \\ 11x_1 - 9x_2 - 6x_3 = 61. \end{cases}$$

Echish. Bu sistemaning kengaytirilgan matritsasini yozamiz

$$A_{keng} = \begin{pmatrix} 3 & 2 & 8 & 4 \\ -5 & 7 & 2 & -43 \\ 11 & -9 & -6 & 61 \end{pmatrix}.$$

Birinchi ustun $\begin{pmatrix} 3 \\ -5 \\ 11 \end{pmatrix}$ ning hamma elementlari noldan farqli. Ularda hohlagan bittasini etakchi element sifatida olish mumkin. Etakchi element sifatida $a_{11} = 3$ olaylik. Birinchi satrni 3 bo‘lamiz

$$A_{keng} \sim \begin{pmatrix} 1 & \frac{2}{3} & \frac{8}{3} & \frac{4}{3} \\ -5 & 7 & 2 & -43 \\ 11 & -9 & -6 & 61 \end{pmatrix}.$$

Birinchi satrni 5 ga ko‘paytirib ikkinchi satrga, (-11) ga ko‘paytirib uchinchi satrga qo‘shamiz

$$A_{keng} \sim \begin{pmatrix} 1 & \frac{2}{3} & \frac{8}{3} & \frac{4}{3} \\ 0 & \frac{31}{3} & \frac{46}{3} & -\frac{109}{3} \\ 0 & -\frac{49}{3} & -\frac{106}{3} & \frac{139}{3} \end{pmatrix}.$$

Ikkinchi satrni $\frac{3}{31}$ ga, uchinchi satni $-\frac{3}{49}$ ga ko‘paytiramiz

$$A_{keng} \sim \begin{pmatrix} 1 & \frac{2}{3} & \frac{8}{3} & \frac{4}{3} \\ 0 & 1 & \frac{46}{31} & -\frac{109}{31} \\ 0 & 1 & \frac{106}{49} & -\frac{139}{49} \end{pmatrix}.$$

Ikkinchi ustundagi ikkinchi elementdan boshqalarini nolga keltiramiz. Buning uchun ikkinchi satrni $-\frac{2}{3}$ ga ko‘paytirib birinchi satrga, -1 ga ko‘paytirib, uchinchi satrga qo‘shamiz

$$A_{keng} \sim \begin{pmatrix} 1 & 0 & \frac{156}{93} & \frac{342}{93} \\ 0 & 1 & \frac{46}{31} & -\frac{109}{31} \\ 0 & 0 & \frac{1032}{1519} & \frac{1032}{1519} \end{pmatrix}.$$

Uchinchi satrni $\frac{1519}{1032}$ ga ko‘paytiramiz

$$A_{keng} \sim \begin{pmatrix} 1 & 0 & \frac{156}{93} & \frac{342}{93} \\ 0 & 1 & \frac{46}{31} & -\frac{109}{31} \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Uchinchi ustundagi birinchi va ikkinchi elementlarni nolga keltiramiz

$$A_{keng} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{186}{93} \\ 0 & 1 & 0 & -\frac{155}{31} \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Demak, berilgan sistema

$$\begin{cases} x_1 = 2, \\ x_2 = -5 \\ x_3 = 1 \end{cases}$$

sistemaga teng kuchli.

Javob. $x_1 = 2$, $x_2 = -5$, $x_3 = 1$.

Bayon qilingan Gauss yoki Jordano-Gauss usullari bir jinslimas chiziqli algebraik tenglamalar sistemasini yechish masalasini hal qiladi. Bir jinsli chiziqli algebraik tenglamalar sistemasini yechishning o‘ziga xos jihatlari mavjud.

3.5. Bir jinsli chiziqli algebraik tenglamalar sistemasining fundamental yechimlari

Bir jinsli chiziqli algebraik tenglamalar sistemasining umumiy ko‘rinishi

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0, \\ \dots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

kabi bo‘ladi. Unga nisbatan qisqalik uchun bir jinsli sistema iborasini ishlatamiz.

Har qanday bir jinsli sistema hech bo‘lmaganda bitta yechimga ega: $x_1 = x_2 = \cdots = x_n = 0$. Bu yechim **trivial** yechim deb ataladi.

– Bir jinsli sistemaning trivial yechimidan tashqari yechimlari mavjudmi? – degan savol paydo bo‘ladi.

Bu savolga javob berish uchun uning asosiy matritsasini yozamiz

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Ravshanki, bu matritsadagi ustunlar soni bir jinsli sistemadagi noma’lumlar soni n ga teng; matritsadagi satrlar soni esa, bir jinsli sistemadagi tenglamalar soni m ga teng.

Matritsa rangi uchun $0 \le rang\ A \le \min\{m,\ n\}$ shartlar bajariladi. Bu tengsizliklarning amaliy ahamiyatga ega xulosalarni keltiramiz. Hech bir holatda na $rang\ A > m$, na $rang\ A > n$ tengsizlik bajarilmaydi. Demak, har doim (sistemadagi tenglamalar soni va noma’lumlarning soni qanday bo‘lishidan qat’iy nazar) $rang\ A \le m$ va $rang\ A \le n$ tengsizliklar bajarilaveradi.

$rang\ A = n$ bo‘lgan hol.

Berilgan fundamental sistema faqatgina trivial echimga ega bo‘lishi uchun $rang\ A = n$ bo‘lishi zarur va etarli.

Bu holda bir jinsli sistemadagi tenglamalar soni noma'lumlar sonidan kam emas ($m \geq n$).

Demak, $m > n$ yoki $m = n$ bo'lishidan qat'iy nazar, $rang\ A = n$ ekanligi bir jinsli sistema faqatgina trivial yechimga egaligini ta'minlaydi.

$rang\ A < n$ bo'lgan hol.

Berilgan bir jinsli sistema trivial yechimdan boshqa yechimga ega bo'lishi uchun $rang\ A < n$ bo'lishi zarur va etarli. Bu holda berilgan sistemaning yechimlari cheksiz ko'p bo'ladi.

Aytaylik, $rang\ A = r$, $r < n$ bo'lsin. Bu degani, bir jinsli sistemaning A matritsasi hech bo'lmaganda bitta noldan farqli r o'lchamli minorga ega. Bunday minor A matritsaning **bazisi** deyiladi. Bazis minorlar bir nechta bo'lishi mumkin.

Tenglamalar sistemasining r ta noma'lumlari oldidagi koeffitsientlar sistema matritsasining bazis minorini hosil qilsa, bu noma'lumlar **bazis**, yoki **asosiy** noma'lumlar deyiladi. Sistemaning qolgan $(n - r)$ ta noma'lumlari **erkli** yoki **asosiymas** noma'lumlar deyiladi.

1) Har bir asosiy noma'lum erkli noma'lumlar orqali chiziqli ifodalanishi mumkin.

2) Erkli noma'lumlarning **har qanday qiymatlari** va asosiy noma'lumlarning ularga mos qiymatlari jamlanmasi bir jinsli tenglamalar sistemasining yechimi bo'ladi. Bu echimlardan $(n - r)$ tasini shunday tanlash mumkinki, bu tanlangan yechimlarning har biri boshqasidan kelib chiqmaydi. Biri boshqasidan kelib chiqmaydigan yechimlar tenglamalar sistemasining **chiziqli erkli yechimlari** deyiladi. Chiziqli erkli yechimlar juda ko'p bo'ladi.

Endi asosiy ta'rifni kiritamiz. Bir jinsli tenglamalar sistemasidagi noma'lumlar soni n ta, uning koeffitsientlaridan tuzilgan A matritsaning rangi r hamda $r < n$ bo'lsin.

Ta'rif. Bir jinsli tenglamalar sistemasining har qanday $(n - r)$ ta chiziqli erkli yechimlaridan iborat to'plam shu sistemaning **fundamental yechimi** deyiladi.

Xossa. Bir jinsli tenglamalar sistemasining har qanday yechimi uning fundamental echimi orqali chiziqli ifodalanadi.

Keltirilgan ta'rif va xossani formulalar orqali tushuntiramiz.

Berilgan ob'ekt

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0, \\ \dots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

bir jinsli chiziqli algebraik tenglamalar sistemasi. Unda n ta noma'lum ishtirok etmoqda. Uning asosiy matritsasi

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

ning rangi r ga teng, ya'ni $rang\ A = r$, va $r < n$ bo'lsin. Unda mavjud bo'lgan noldan farqli, r-tartibli minorlaridan birortasi

$$M = \begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_r} \\ a_{i_2 j_1} & a_{i_2 j_2} & \cdots & a_{i_2 j_r} \\ \cdots & \cdots & \ddots & \cdots \\ a_{i_r j_1} & a_{i_r j_2} & \cdots & a_{i_r j_r} \end{vmatrix}$$

bo‘lsin. u holda $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$ lar asosiy noma’lumlar, $x_{j_{r+1}}, x_{j_{r+2}}, \ldots, x_{j_n}$ lar erkli noma’lumlar bo‘ladi. Erkli noma’lumlarning turlicha $x_{j_{r+1}} = \alpha_{j_{r+1}}^k$, $x_{j_{r+2}} = \alpha_{j_{r+2}}^k$, …, $x_{j_n} = \alpha_{j_n}^k$ qiymatlariga asosiy o‘zgaruvchilarning mos $x_{j_1} = \alpha_{j_1}^k$, $x_{j_2} = \alpha_{j_2}^k$, …, $x_{j_r} = \alpha_{j_r}^k$ qiymatlari hisoblangan bo‘lsin, $k = 1, 2, \ldots, n - r$. U holda

$$\left\{ \begin{pmatrix} \alpha_1^1 \\ \alpha_2^1 \\ \vdots \\ \alpha_n^1 \end{pmatrix}, \begin{pmatrix} \alpha_1^2 \\ \alpha_2^2 \\ \vdots \\ \alpha_n^2 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_1^{n-r} \\ \alpha_2^{n-r} \\ \vdots \\ \alpha_n^{n-r} \end{pmatrix} \right\}$$

to‘plam berilgan sistemaning **fundamental yechimi** bo‘ladi. Berilgan sistemaning **umumiy yechimi** esa

$$\left\{ C_1 \cdot \begin{pmatrix} \alpha_1^1 \\ \alpha_2^1 \\ \vdots \\ \alpha_n^1 \end{pmatrix} + C_2 \cdot \begin{pmatrix} \alpha_1^2 \\ \alpha_2^2 \\ \vdots \\ \alpha_n^2 \end{pmatrix} + \cdots + C_{n-r} \cdot \begin{pmatrix} \alpha_1^{n-r} \\ \alpha_2^{n-r} \\ \vdots \\ \alpha_n^{n-r} \end{pmatrix} : \quad C_1, C_2, \ldots, C_{n-r} \in \mathbb{R} \right\}$$

to‘plamdan iborat.

Demak, bir jinsli chiziqli algebraik tenglamalar sistemasining umumiy yechimini topish uchun uning fundamental echimlaridan birontasini ko‘rsatish kifoya ekan.

Soddalik uchun fundamental va umumiy yechimlar

$$\begin{pmatrix} \alpha_1^1 \\ \alpha_2^1 \\ \vdots \\ \alpha_n^1 \end{pmatrix}, \begin{pmatrix} \alpha_1^2 \\ \alpha_2^2 \\ \vdots \\ \alpha_n^2 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_1^{n-r} \\ \alpha_2^{n-r} \\ \vdots \\ \alpha_n^{n-r} \end{pmatrix}$$

va

$$C_1 \cdot \begin{pmatrix} \alpha_1^1 \\ \alpha_2^1 \\ \vdots \\ \alpha_n^1 \end{pmatrix} + C_2 \cdot \begin{pmatrix} \alpha_1^2 \\ \alpha_2^2 \\ \vdots \\ \alpha_n^2 \end{pmatrix} + \cdots + C_{n-r} \cdot \begin{pmatrix} \alpha_1^{n-r} \\ \alpha_2^{n-r} \\ \vdots \\ \alpha_n^{n-r} \end{pmatrix}$$

kabi yoziladi, bu erda $C_1, C_2, \ldots, C_{n-r} \in \mathbb{R}$ – ixtiyoriy sonlar.

Yuqorida bayon qilinganlarni yaqqol tushuntirish maqsadida quyidagi misolni keltiramiz.

3.5.1.-misol. Bir jinsli tenglamalar sistemasini eching. Uning fundamental va umuiy yechimlarini ko‘rsating.

$$\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 - 4x_5 = 0, \\ -2x_1 + 5x_2 - 11x_3 + 11x_5 = 0, \\ 2x_1 - 5x_2 + 12x_3 - 4x_4 - 9x_5 = 0, \\ x_1 - x_2 - 2x_3 + 3x_4 - x_5 = 0. \end{cases}$$

Yechish. Berilgan sistemaning asosiy matritsasini tuzamiz va tuzilgan matritsaning rangini topamiz.

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 & -4 \\ -2 & 5 & -11 & 0 & 11 \\ 2 & -5 & 12 & -4 & -9 \\ 1 & -1 & -2 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 1 & -4 \\ 0 & 1 & -5 & 2 & 3 \\ 0 & 1 & -6 & 6 & 1 \\ 0 & -1 & 5 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 1 & -4 \\ 0 & 1 & -5 & 2 & 3 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow rang\, A = 3\,.$$

Demak, A matritsada noldan farqli, **o‘lchami 3 ga teng minor** mavjud ekan. Bunday minorlardan birontasini, aytaylik,

$$\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{vmatrix}$$

minorni tayinlaylik. Bu minorning elementlari berilgan bir jinsli tenglamalar sistemasiga teng kuchli ushbu

$$\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 - 4x_5 = 0, \\ \qquad x_2 - 5x_3 + 2x_4 + 3x_5 = 0, \\ \qquad\qquad x_3 - 4x_4 + 2x_5 = 0 \end{cases}$$

sistema tenglamalarining x_1, x_2 va x_3 noma’lumlari oldida turgan koeffitsientlardir. Demak, bu sistema uchun x_1, x_2 va x_3 lar **asosiy** noma’lumlar bo‘ladi; x_4, x_5 lar esa **erkli** noma’lumlar bo‘ladi.

Endi x_1, x_2 va x_3 larni x_4, x_5 lar orqali ifodalaymiz:

$$\begin{cases} x_1 = 23x_4 - 16x_5, \\ x_2 = 18x_4 - 13x_5, \\ x_3 = 4x_4 - 2x_5 \end{cases}$$

Chiziqli erkli echimlarni topamiz. Buning uchun erkli o‘zgaruvchilarga ($n - r = 5 - 3 = 2$ bo‘lgani uchun) ikkita turli xil ixtiyoriy qiymatlar beriladi. Aytaylik, birinchi holda $x_4 = 0$ va $x_5 = -1$ bo‘lsin. Bu holda

$$x_1 = 16,$$
$$x_2 = 13,$$
$$x_3 = 2$$

bo‘lib, $(16,\ 13,\ 2,\ 0,\ -1)$ jamlanma berilgan sistemaning bitta yechimi bo‘ladi (**tekshirib ko‘ring!**).

Ikkinchi holda, aytaylik, $x_4 = 1$ va $x_5 = 0$ bo‘lsin. Bu holda

$$x_1 = 23,$$
$$x_2 = 18,$$
$$x_3 = 4$$

bo‘lib, $(23,\ 18,\ 4,\ 1,\ 0)$ jamlanma berilgan sistemaning boshqa bitta yechimi bo‘ladi.

Bu ikkita $(16,\ 13,\ 2,\ 0,\ -1)$ va $(23,\ 18,\ 4,\ 1,\ 0)$ yechimlar chiziqli erkli bo‘lishini

$$\begin{pmatrix} 16 & 13 & 2 & 0 & -1 \\ 23 & 18 & 4 & 1 & 0 \end{pmatrix}$$

matritsaning rangi 2 ga tengligi orqali ko‘rsatish mumkin. Shuning uchun bu yechimlardan tuzilgan $\{(16,\ 13,\ 2,\ 0,\ -1),\ (23,\ 18,\ 4,\ 1,\ 0)\}$ to‘plam qaralayotgan sistemaning **fundamental yechimi** bo‘ladi.

Demak, fundamental yechimga ega bo‘ldik:

$$\begin{pmatrix} 16 \\ 13 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 23 \\ 18 \\ 4 \\ 1 \\ 0 \end{pmatrix}.$$

Berilgan misoldagi bir jinsli sitemaning **umumiy yechimi**

$$C_1 \cdot \begin{pmatrix} 16 \\ 13 \\ 2 \\ 0 \\ -1 \end{pmatrix} + C_2 \cdot \begin{pmatrix} 23 \\ 18 \\ 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 16C_1 + 23C_2 \\ 13C_1 + 18C_2 \\ 2C_1 + 4C_2 \\ C_2 \\ -C_1 \end{pmatrix}.$$

ko‘rinishda bo‘ladi. Bu erda C_1, C_2 – ixtiyoriy sonlar.

Mustahkamlash uchun mashqlar

Chiziqli algebraik tenglamalar sistemalariga oid masalalar

1. Tenglamalar sistemasining asosiy matritsasi, kengaytirilgan matritsasi, noma’lumlar matritsasi, ozod hadlar matritsasini yozing

1) $\begin{cases} 2x_1 - x_2 + 7x_3 + x_4 - 3x_5 = -6, \\ 3x_1 + 2x_2 - 8x_3 + 5x_4 + x_5 = -8, \\ 7x_1 + x_2 + 4x_3 - 2x_4 - 6x_5 = -18, \end{cases}$

2) $\begin{cases} 7x_1 - x_2 - 6x_3 = -14, \\ 5x_1 + 2x_2 + 3x_3 = 0, \\ 2x_1 + x_2 + 8x_3 = 7, \\ 3x_1 + 9x_2 + 4x_3 = 10. \end{cases}$

3) $\begin{cases} 2x_1 - x_3 + x_4 - 3x_5 = -9, \\ 2x_2 - 8x_3 + 5x_4 + x_5 = -4, \\ 7x_1 + x_2 + 4x_3 - 2x_4 = -2, \end{cases}$

4) $\begin{cases} 7x_1 - x_2 = -8, \\ 5x_1 + 2x_2 + 3x_3 = 0, \\ x_2 + 8x_3 = 9, \\ x_1 + 4x_3 = 3. \end{cases}$

5) $\begin{cases} 2x_1 - x_2 + 3x_3 = 0, \\ 8x_3 + 5x_4 + x_5 = 10, \\ x_2 + 4x_3 - 2x_4 = 5, \end{cases}$

6) $\begin{cases} -x_2 - 6x_3 = -7, \\ 5x_1 + 2x_2 = -3, \\ 2x_1 + 8x_3 = 6, \\ 3x_1 + 9x_2 + 4x_3 = 10. \end{cases}$

2. Tenglamalar sistemasini tekshiring

1) $\begin{cases} 2x_1 - x_2 + 7x_3 + x_4 - 3x_5 = -6, \\ 3x_1 + 2x_2 - 8x_3 + 5x_4 + x_5 = -8, \\ 7x_1 + x_2 + 4x_3 - 2x_4 - 6x_5 = -18, \end{cases}$

2) $\begin{cases} 7x_1 - x_2 - 6x_3 = -14, \\ 5x_1 + 2x_2 + 3x_3 = 0, \\ 2x_1 + x_2 + 8x_3 = 7, \\ 3x_1 + 9x_2 + 4x_3 = 10, \end{cases}$

3) $\begin{cases} 2x_1 - x_3 + x_4 - 3x_5 = -9, \\ 2x_2 - 8x_3 + 5x_4 + x_5 = -4, \\ 7x_1 + x_2 + 4x_3 - 2x_4 = -2, \end{cases}$

4) $\begin{cases} 7x_1 - x_2 = -8, \\ 5x_1 + 2x_2 + 3x_3 = 0, \\ x_2 + 8x_3 = 9, \\ x_1 + 4x_3 = 3, \end{cases}$

5) $\begin{cases} 2x_1 - x_2 + 3x_3 = 0, \\ 8x_3 + 5x_4 + x_5 = 10, \\ x_2 + 4x_3 - 2x_4 = 5, \end{cases}$

6) $\begin{cases} -x_2 - 6x_3 = -7, \\ 5x_1 + 2x_2 = -3, \\ 2x_1 + 8x_3 = 6, \\ 3x_1 + 9x_2 + 4x_3 = 10. \end{cases}$

3. Berilgan tenglamalar sistemasini matritsali ko‘rinishda yozing

1) $\begin{cases} 2x_1 - x_2 + 7x_3 + x_4 - 3x_5 = -6, \\ 3x_1 + 2x_2 - 8x_3 + 5x_4 + x_5 = -8, \\ 7x_1 + x_2 + 4x_3 - 2x_4 - 6x_5 = -18, \end{cases}$

2) $\begin{cases} 7x_1 - x_2 - 6x_3 = -14, \\ 5x_1 + 2x_2 + 3x_3 = 0, \\ 2x_1 + x_2 + 8x_3 = 7, \\ 3x_1 + 9x_2 + 4x_3 = 10, \end{cases}$

3) $\begin{cases} 2x_1 - x_3 + x_4 - 3x_5 = -9, \\ 2x_2 - 8x_3 + 5x_4 + x_5 = -4, \\ 7x_1 + x_2 + 4x_3 - 2x_4 = -2, \end{cases}$

4) $\begin{cases} 7x_1 - x_2 = -8, \\ 5x_1 + 2x_2 + 3x_3 = 0, \\ x_2 + 8x_3 = 9, \\ x_1 + 4x_3 = 3, \end{cases}$

5) $\begin{cases} 2x_1 - x_2 + 3x_3 = 0, \\ 8x_3 + 5x_4 + x_5 = 10, \\ x_2 + 4x_3 - 2x_4 = 5, \end{cases}$ 6) $\begin{cases} -x_2 - 6x_3 = -7, \\ 5x_1 + 2x_2 = -3, \\ 2x_1 + 8x_3 = 6, \\ 3x_1 + 9x_2 + 4x_3 = 10. \end{cases}$

4. Tenglamalar sistemasini Kramer usulidan foydalanib yeching

1) $\begin{cases} 2x_1 - x_2 - 3x_3 = -6, \\ 3x_1 + 2x_2 + x_3 = 0, \\ 7x_1 + x_2 - 6x_3 = -12, \end{cases}$ 2) $\begin{cases} 5x_1 + 2x_2 + 3x_3 = -1, \\ 2x_1 + x_2 + 8x_3 = -14, \\ 3x_1 + 9x_2 + 4x_3 = -5, \end{cases}$

3) $\begin{cases} -x_1 + 7x_2 + x_3 = 6, \\ 2x_1 - 8x_2 + 5x_3 = -6, \\ x_1 + 4x_2 - 2x_3 = 5, \end{cases}$ 4) $\begin{cases} 7x_1 - x_2 = 13, \\ x_2 + 8x_3 = 1, \\ x_1 + 4x_3 = 2, \end{cases}$

5) $\begin{cases} 7x_1 + x_2 - 3x_3 = 1, \\ 8x_1 + 5x_2 + x_3 = 10, \\ x_1 + 4x_2 - 2x_3 = -3, \end{cases}$ 6) $\begin{cases} -x_2 - 6x_3 = -5 \\ 5x_1 + 2x_2 = -2, \\ 2x_1 + 8x_3 = 8. \end{cases}$

5. Tenglamalar sistemasini teskari matritsalar usulidan foydalanib yeching

1) $\begin{cases} 2x_1 - x_2 - 3x_3 = -6, \\ 3x_1 + 2x_2 + x_3 = 0, \\ 7x_1 + x_2 - 6x_3 = -12, \end{cases}$ 2) $\begin{cases} 5x_1 + 2x_2 + 3x_3 = -1, \\ 2x_1 + x_2 + 8x_3 = -14, \\ 3x_1 + 9x_2 + 4x_3 = -5, \end{cases}$

3) $\begin{cases} -x_1 + 7x_2 + x_3 = 6, \\ 2x_1 - 8x_2 + 5x_3 = -6, \\ x_1 + 4x_2 - 2x_3 = 5, \end{cases}$ 4) $\begin{cases} 7x_1 - x_2 = 13, \\ x_2 + 8x_3 = 1, \\ x_1 + 4x_3 = 2, \end{cases}$

5) $\begin{cases} 7x_1 + x_2 - 3x_3 = 1, \\ 8x_1 + 5x_2 + x_3 = 10, \\ x_1 + 4x_2 - 2x_3 = -3, \end{cases}$ 6) $\begin{cases} -x_2 - 6x_3 = -5 \\ 5x_1 + 2x_2 = -2, \\ 2x_1 + 8x_3 = 8. \end{cases}$

6. Tenglamalar sistemasini Guass usulidan foydalanib yeching

1) $\begin{cases} 2x_1 - x_2 - 3x_3 = -6, \\ 3x_1 + 2x_2 + x_3 = 0, \\ 7x_1 + x_2 - 6x_3 = -12, \end{cases}$ 2) $\begin{cases} 5x_1 + 2x_2 + 3x_3 = -1, \\ 2x_1 + x_2 + 8x_3 = -14, \\ 3x_1 + 9x_2 + 4x_3 = -5, \end{cases}$

3) $\begin{cases} -x_1 + 7x_2 + x_3 = 6, \\ 2x_1 - 8x_2 + 5x_3 = -6, \\ x_1 + 4x_2 - 2x_3 = 5, \end{cases}$ 4) $\begin{cases} 7x_1 - x_2 = 13, \\ x_2 + 8x_3 = 1, \\ x_1 + 4x_3 = 2, \end{cases}$

5) $\begin{cases} 7x_1 + x_2 - 3x_3 = 1, \\ 8x_1 + 5x_2 + x_3 = 10, \\ x_1 + 4x_2 - 2x_3 = -3, \end{cases}$ 6) $\begin{cases} -x_2 - 6x_3 = -5 \\ 5x_1 + 2x_2 = -2, \\ 2x_1 + 8x_3 = 8. \end{cases}$

7. Tenglamalar sistemasini Jordano-Guass usulidan foydalanib yeching

1) $\begin{cases} 2x_1 - x_2 - 3x_3 = -6, \\ 3x_1 + 2x_2 + x_3 = 0, \\ 7x_1 + x_2 - 6x_3 = -12, \end{cases}$ 2) $\begin{cases} 5x_1 + 2x_2 + 3x_3 = -1, \\ 2x_1 + x_2 + 8x_3 = -14, \\ 3x_1 + 9x_2 + 4x_3 = -5, \end{cases}$

3) $\begin{cases} -x_1 + 7x_2 + x_3 = 6, \\ 2x_1 - 8x_2 + 5x_3 = -6, \\ x_1 + 4x_2 - 2x_3 = 5, \end{cases}$ 4) $\begin{cases} 7x_1 - x_2 = 13, \\ x_2 + 8x_3 = 1, \\ x_1 + 4x_3 = 2, \end{cases}$

5) $\begin{cases} 7x_1 + x_2 - 3x_3 = 1, \\ 8x_1 + 5x_2 + x_3 = 10, \\ x_1 + 4x_2 - 2x_3 = -3, \end{cases}$ 6) $\begin{cases} -x_2 - 6x_3 = -5 \\ 5x_1 + 2x_2 = -2, \\ 2x_1 + 8x_3 = 8. \end{cases}$

8. Bir jinsli tenglamalar sistemasining ko‘rsatilgan shartlarni qanoatlantiruvchi fundamental yechimlarini ko‘rsating.

$$\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 - 4x_5 = 0, \\ -2x_1 + 5x_2 - 11x_3 + 11x_5 = 0, \\ 2x_1 - 5x_2 + 12x_3 - 4x_4 - 9x_5 = 0, \\ x_1 - x_2 - 2x_3 + 3x_4 - x_5 = 0. \end{cases}$$

1) $x_4 = 0$ va $x_5 = 1$; $x_4 = 1$ va $x_5 = 1$, 2) $x_4 = 1$ va $x_5 = 2$; $x_4 = 1$ va $x_5 = 0$,
3) $x_4 = 1$ va $x_5 = 0$; $x_4 = 1$ va $x_5 = 2$, 4) $x_4 = 2$ va $x_5 = 2$; $x_4 = 0$ va $x_5 = -1$,
5) $x_4 = 1$ va $x_5 = 2$; $x_4 = 2$ va $x_5 = 2$, 6) $x_4 = 2$ va $x_5 = 1$; $x_4 = 0$ va $x_5 = -1$.

O`z-o`zini tekshirish uchun savollar

1. Chiziqli algebraik tenglamalar sistemasi deb nimaga aytiladi?
2. Chiziqli algebraik tenglamalar sistemasining matritsali shakli qanday yoziladi?
3. Chiziqli algebraik tenglamalar sistemasining birgalikda bo`lishini ifodalovchi teoremani ayting.
4. Chiziqli algebraik tenglamalar sistemasini yechishning qanday usullarini bilasiz?
5. Chiziqli algebraik tenglamalar sistemasini yechishning Kramer usulini tushuntiring.
6. Chiziqli algebraik tenglamalar sistemasini yechishning teskari matritsa usulini tushuntiring.
7. Chiziqli algebraik tenglamalar sistemasini yechishning Gauss usulini tushuntiring.
8. Chiziqli algebraik tenglamalar sistemasini yechishning Gauss-Jordan usulini tushuntiring.
9. Bir jinsli chiziqli algebraik tenglamalar sistemasining fundamental yechimlari qanday topiladi?

O`z-o`zini tekshirish uchun testlar

Chiziqli algebraik tenglamalar sistemalariga oid testlar

1. $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ chiziqli algebraik tenglamalar sistemasining yechimi deb nimaga aytildi?

a) x_1, x_2, …, x_n noma'lumlarning o'rniga qo'yganda sistemaning har bir tenglamasini ayniyatga aylantiruvchi tartiblangan $(\alpha_1, \alpha_2, \dots, \alpha_n)$ sonlar jamlanmasiga aytiladi.
b) a_{ij} sonlar sistemaning yechimi deyiladi, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.
c) b_j sonlar esa sistemaning yechimi deyiladi, $j = 1, 2, \dots, m$.
d) Tenglamalar sistemasining yechimi deb, x_1, x_2, …, x_n larga aytiladi.

2. $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ chiziqli algebraik tenglamalar sistemasida x_1, x_2, …, x_n lar … deyiladi.

a) noma'lumlar b) ozod hadlari
c) koeffitsientlari d) yechimi

3. $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ chiziqli algebraik tenglamalar sistemasida a_{ij} $(i = 1,\ 2,\ \dots,\ n, j = 1,\ 2, \dots, m.)$ sonlar sistemaning … deyiladi.

a) noma'lumlar b) ozod hadlari
c) koeffitsientlari d) yechimi

4. $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ chiziqli algebraik tenglamalar sistemasida b_j $(j = 1,\ 2, \dots, m.)$ sonlar sistemaning … deyiladi.

a) noma'lumlar b) ozod hadlari
c) koeffitsientlari d) yechimi

5. Agar $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ chiziqli algebraik tenglamalar sistemasida $b_j = 0$ $(j = 1,\ 2, \dots, m.)$ bo'lsa, u holda bunday sistema … deyiladi.

a) bir jinslimas b) bir jinsli
c) aniq d) aniqmas

6. Agar $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ chiziqli algebraik tenglamalar sistemasida kamida bitta ozod had noldan farqli bo'lsa, u holda bunday sistema … deyiladi.

a) bir jinslimas b) bir jinsli c) aniq d) aniqmas

7. Bironta ham yechimga ega bo'lmagan chiziqli algebraik tenglamalar sistemasiga … tenglamalar sistemasi deyiladi.

a) birgalikda bo'lmagan b) bir jinsli c) aniq d) aniqmas

8. Yagona yechimga ega bo'lgan chiziqli algebraik tenglamalar sistemasiga … tenglamalar sistemasi deyiladi.

a) birgalikda bo'lmagan b) bir jinsli c) aniq d) aniqmas

9. Cheksiz ko'p yechimga ega bo'lgan chiziqli algebraik tenglamalar sistemasiga … tenglamalar sistemasi deyiladi.

a) birgalikda bo'lmagan b) bir jinsli c) aniq d) aniqmas

10. Chiziqli tenglamalar sistemasining noma`lumlar oldidagi koeffitsientlardan tuzilgan matritsa...matritsa deb ataladi

a) asosiy b) kengaytirilgan c) birgalikda d) uchburchak

11. Chiziqli tenglamalar sistemasining asosiy matritsasi ozod hadlar ustuni bilan birgalikda...matritsa deb ataladi.

a) asosiy b) kengaytirilgan c) birgalikda d) uchburchak

12. Bir jinslichiziqli tenglamalar sistemasi har doim...

a) kamida bitta yechimga ega b) yechimga ega emas
c) ikkita yechimga ega d) kamida ikkita yechimga ega

13. n to‘rtdan katta bo‘lganda n noma`lumli n ta chiziqli tenglamalar sistemasini quyidagi usullardan qaysi birida yechish qulayroq.
a)Gauss b) Kramer c) O‘rniga qo‘yish usuli d) Matritsalar usuli

14. $AX = B$ chiziqli tenglamalar sistemasining matritsa ko‘rinishdagi yechimini ko‘rsating.
a) $X = A^{-1}B$ b) $X = AB$ c) $X = A^{-1}A$ d) $X = A/B$

15. $\begin{cases} 2x - y = -1 \\ x + 2y - z = -2 \\ y + z = -2 \end{cases}$ teglamalar sistemasini yeching.
a) $x = y = z = -1$ b) $x = y = z = 0$
c) $x = y = z = -2$ d) $x = y = z = 1$

16. $\begin{cases} 6x + 2y = 3 \\ 3x + y = 2 \end{cases}$ sistemani tekshiring.
a) Birgalikda emas b) Uchta yechimga ega
c) Ikkita yechimga ega d)Birgalikda

17. Agar chiziqli algebraik tenglamalar sistemasida $rang\ A = rang\ A_{keng}$ bo‘lsa, u holda … sistema bo‘ladi.
a) birgalikda bo‘lgan b) birgalikda bo‘lmagan
c) ikkita yechimga ega d) uchta yechimga ega

18. Agar chiziqli algebraik tenglamalar sistemasida $rang\ A \neq rang\ A_{keng}$ bo‘lsa, u holda … sistema bo‘ladi.
a) birgalikda bo‘lgan b) birgalikda bo‘lmagan
c) ikkita yechimga ega d) cheksiz ko‘p yechimga ega

19. Agar chiziqli algebraik tenglamalar sistemasida $rang\ A = rang\ A_{keng} < n$ bo‘lsa, u holda … sistema bo‘ladi.
a) uchta yechimga ega bo‘lgan b) birgalikda bo‘lmagan
c) ikkita yechimga ega d) cheksiz ko‘p yechimga ega

20. Agar chiziqli algebraik tenglamalar sistemasida $rang\ A = rang\ A_{keng} = n$ bo‘lsa, u holda … sistema bo‘ladi.
a) aniq b) birgalikda bo‘lmagan
c) ikkita yechimga ega d) cheksiz ko‘p yechimga ega

21. Bir jinslimas chiziqli algebraik tenglamalar sistemasini yechishning qanday usullar bor?
a) Kramer usuli, teskari matritsa usuli, Gauss usuli va Gauss-Jordan
b) Birgalikda bo‘lmagan, aniq, aniqmas
c) Birgalikda bo‘lgan, aniq, aniqmas, bir jinsli
d) Bir jinsli bo‘lmagan, aniqmas, bir jinsli

22. Chiziqli algebraik tenglamalar sistemasini yechishning Kramer usuli qo‘llash uchun nimalar talab etiladi?
a) sistemasidagi tenglamalar soni bilan noma’lumlar soni teng hamda asosiy matritsaning determinanti noldan farqli bo‘lishi talab etiladi.
b) sistemasidagi tenglamalar soni bilan noma’lumlar soni teng bo‘lishi talab etiladi.

c) sistemasidagi tenglamalar soni bilan noma'lumlar soni teng hamda asosiy matritsaning determinanti nolga teng bo'lishi talab etiladi.
d) sistemasidagi asosiy matritsaning determinanti nolga teng bo'lishi talab etiladi.
23. Chiziqli algebraik tenglamalar sistemasini yechishning teskari matritsa usuli qo'llash uchun nimalar talab etiladi?
a) sistemasidagi tenglamalar soni bilan noma'lumlar soni teng hamda asosiy matritsaning determinanti noldan farqli bo'lishi talab etiladi.
b) sistemasidagi tenglamalar soni bilan noma'lumlar soni teng bo'lishi talab etiladi.
c) sistemasidagi tenglamalar soni bilan noma'lumlar soni teng hamda asosiy matritsaning determinanti nolga teng bo'lishi talab etiladi.
d) sistemasidagi asosiy matritsaning determinanti nolga teng bo'lishi talab etiladi.

24. $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$ sistemaning asosiy matritsasini ko'rsating.

a) $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ b) $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

c) $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ d) $A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$

25. $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$ sistemaning noma'lumlardan tuzilgan matritsasini ko'rsating.

a) $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ b) $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

c) $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ d) $A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$

26. $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \dots, \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$ sistemaning ozod hadlardan tuzilgan matritsasini ko'rsating.

a) $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ b) $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

c) $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ d) $A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$

27. Teskari matritsani topish formulasini ko‘rsating.

a) $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ b) $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

c) $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ d) $A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$

28. $\begin{cases} 2x - y + z = 2 \\ x + 2y - z = 2 \\ -4x + y + z = -2 \end{cases}$ teglamalar sistemasini yeching.

a) $x = y = z = -1$ b) $x = y = z = 0$
c) $x = y = z = -2$ d) $x = y = z = 1$

29. $\begin{cases} x + 2y = -3 \\ x - y = 0 \end{cases}$ teglamalar sistemasini yeching.

a) $x = y = -1$ b) $x = y = 0$
c) $x = y = -2$ d) $x = y = 1$

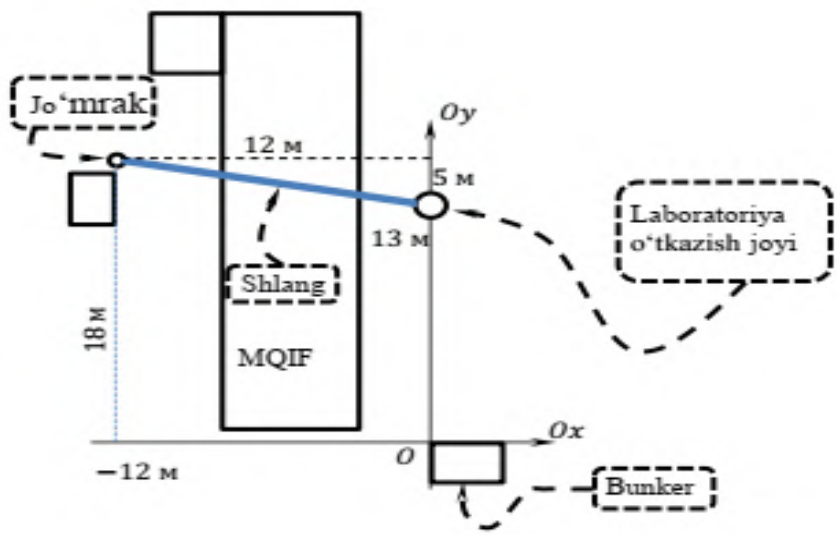
30. $\begin{cases} 2x - y + z = 2 \\ 4x - 2y + 2z = 2 \\ -4x + 2y - 2z = -1 \end{cases}$ sistemani tekshiring.

a) Birgalikda emas b) Uchta yechimga ega
c) Ikkita yechimga ega d)Birgalikda

4-§. KOORDINATALAR SISTEMALARI

Koordinatalar sistemalariga oid masalalar

1. Toshkent arxitektura qurilish instituti Muhandislik qurilish infrastrukturasi fakulteti (MQIF) binosi orqa tomonida laboratoriya ishi o‘tkazilishi rejalashtirildi. Laboratoriya o‘tkazishda ko‘p suv talab etiladi. Suv jo‘mragi binoning old tomonida joylashgan. Shlangni bino derazalari orqali o‘tkazishga ruxsat berildi. Laboratoriya o‘tkaziladigan joy “Bunker” kanselyariya mollari do‘konidan janubga qarab 13 metr masofa bo‘lib, suv jo‘mragi “Bunker” dan sharqqa 12 metr, janubga 18 metr masofada joylashgan. Talab etiladigan shlang uzunligini toping.



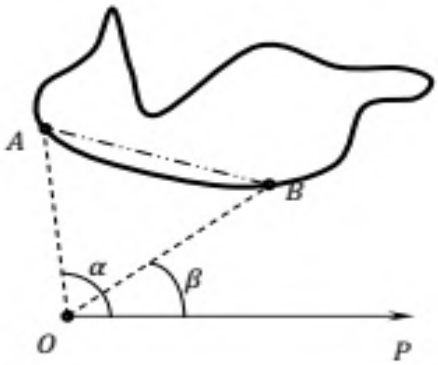
Yechish. MQIF binosi yonida joylashgan “Bunker” kanselyariya mollari do‘konini koordinata boshi deb olib, janubga yo‘nalgan o‘qni Oy o‘q, g‘arbga yo‘nalgan o‘qni Ox o‘q, birlik kesma deb 1 m ni olamiz.

Hosil bo‘lgan Dekart koordinatalar sistemasida jo‘mrak koodinatalari $J(-12, 18)$, laboratoriya o‘tkazish joyi koordinatalari $L(0, 13)$ bo‘ladi. JL kesma uzunligi topilsa, masala echilgan bo‘ladi.

$$AB = \sqrt{\left(0-(-12)\right)^2 + (18-13)^2} = \sqrt{169} = 13\ m.$$

Javob: 13 metr.

Izoh. Agar shlang bino atrofi bo‘ylab o‘tkazilsa, $18 + 12 + 13 = 43\ m$ talab etilar edi.



2. Talaba geodezik o‘lchovlar natijasida rasmda tasvirlangan shakldagi ko‘lning qirg‘og‘idagi A va B nuqtalari uchun quyidagi natijalarni oldi: $\alpha = 120^0$, $\beta = 60^0$, $OA = 120$ m, $OB = 150$ m. Bu ikkita nuqtalar orasidagi masofani toping.

Yechish. AOB uchburchakni chizamiz. Kosinuslar teoremasidan

$$AB = \sqrt{OA^2 + OB^2 - 2OA \cdot OB\cos(\alpha - \beta)} =$$
$$= \sqrt{120^2 + 150^2 - 2 \cdot 120 \cdot 150\cos 60^0} =$$
$$\sqrt{18900} \approx 137{,}5\ m.$$

Javob: 137,5 metr.

Izoh. Masala aslida qutb koordinatalari sistemasi orqali yechildi. O qutb, OA va OB masofalar mos ravishda A va B nuqtalarning radiuslari, α va β burchaklar mos A va B nuqtaning qutb burchaklari deb yuritiladi.

A va B nuqtalar orqali o‘tuvchi to‘g‘ri chiziq ko‘lning suvli qismidan o‘tgani uchun, ular orasidagi masofani quruqlik orqali o‘lchash imkoni yo‘q.

3. Toshkent teleminorasi va Interkontinental mehmonxonasining eng yuqori nuqtalari orasidagi masofani toping. Toshkent teleminorasining balandligi 375 m, Interkontinental mehmonxonasining eng yuqori nuqtasi yer sathidan 215 m balandlikda joylashgan. Bunda kichik xalqa yo‘li bilan Sharof Rashidov ko‘chalari kesishmasini qutb, Kichik xalqa yo‘lidan Amir Temur ko‘chasi tarafga yo‘nalishni musbat yo‘nalish deb olingan. O‘lchashlar natijasiga ko‘ra mehmonxona cho‘qqisining silindrik koordinatalari $A(300,\ 30^0,\ 215)$, Teleminora cho‘qqisining silindrik koordinatalari $B(1000,\ 60^0,\ 375)$.

Yechish. Ushbu formuladan foydalanamiz

$$\rho(A,B) = AB = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\varphi_B - \varphi_A) + (z_B - z_A)^2}.$$

Bu yerda (r_A, φ_A, z_A) va (r_B, φ_B, z_B) uchliklar mos ravishda A va B nuqtalarning silindrik koordinatalari.

$$\rho(A,B) = AB = \sqrt{300^2 + 1000^2 - 2 \cdot 300 \cdot 1000 \cos 30^0 + 160^2} \approx 1800.$$

Javob: Toshkent teleminorasi va Interkontinental mehmonxonasi cho‘qqilari orasidagi masofa ≈ 1800 metr ekan.

4. Koordinatalar sistemalari (nazariy qism)

Yon atrofimizda joylashgan ob’ektlarning, sodir bo‘layotgan voqea, hodisalarning bizga nisbatan joylashuvi, vaziyati, fazoviy o‘rni haqidagi ma’lumotlar ko‘plab talab etiladi. Bu talablarni qondirish maqsadida o‘rganilayotgan sirt yoki fazoga **koordinatalar sistemalari** kiritiladi. Dastlab **qutb koordinatalar va sferik koordinatalar sistemalari** keng qo‘lanilgan.

Qutb koordinatalar sistemasida tekislikdagi har bir nuqta ikkita son – qutb radius va qutb burchak – orqali aniqlanadi. Burchak va radius tushunchalari eraamizdan avvalgi birinchi ming yillikdayoq ma’lum bo‘lgan.

Keyinchalik to‘g‘ri burchakli **koordinatalar sistemasi** qo‘llanila boshlangan. To‘g‘ri burchakli koordinatalar sistemasini Dekart koordinatalar sistemasi ham deyishadi (Rene Dekart – fransuzcha *René Descartes*, lotincha *Renatus Cartesius* – fransuz matematigi (31.03.1596 – 11.02.1650). U 1637 yilda “Metod haqida mulohaza” asarida koordinatalar sistemasini kiritadi).

Koordinatalar sistemasi kiritilgach, jismlarni tavsiflashning ajoyib usuli – ularni tenglamalar yoki tengsizliklar orqali ifodalash usuli paydo bo‘ldi.

4.1. Dekart koordinatalar sistemasi

Biror to‘g‘ri chiziqda **hisob boshi** (kuzatuv nuqtasi, O nuqta) **ni** tanlaymiz. O nuqtadan to‘g‘ri chiziq bo‘ylab ikki tomonga yo‘nalish mumkin. Bu yo‘nalishlardan

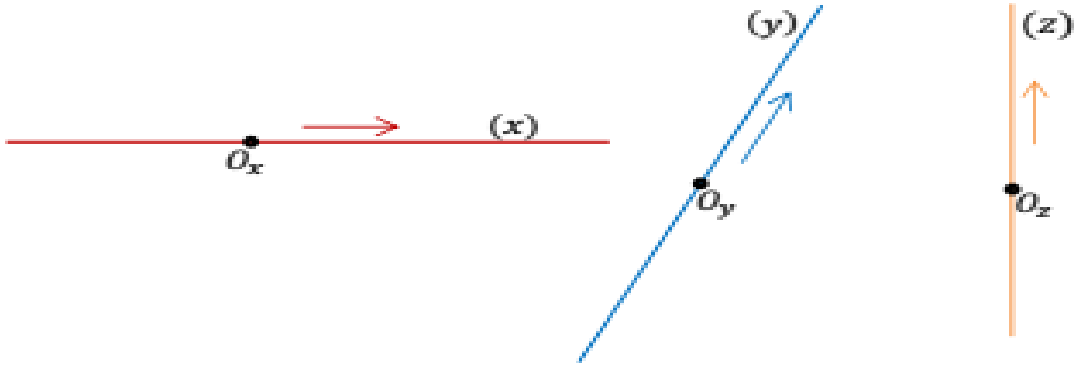
bittasini musbat yo‘nalish deb olish mumkin. Biz quyidagilarga kelishib olamiz. Agarda hisob boshi tayinlangan to‘g‘ri chiziq:

(x) kuzatuvchining qo‘llari yoyilgan holatiga parallel bo‘lsa, undagi (kuzatuvchiga nisbatan) chapdan o‘ngga yo‘nalishni;

(y) kuzatuvchining nigohiga parallel bo‘lsa, undagi (kuzatuvchiga nisbatan) orqadan oldga yo‘nalishni;

(z) kuzatuvchining tik holatiga parallel bo‘lsa, undagi pastdan tepaga yo‘nalishni

musbat yo‘nalish deb qabul qilamiz.



Yo‘nalish kelishib olingach, to‘g‘ri chiziqda birlik kesma ko‘rsatiladi.

Hisob boshi, musbat yo‘nalishi va birlik kesmasi tayinlangan to‘g‘ri chiziq **koordinata o‘qi** deb yuritiladi. Tayinlangan hisob boshi **koordinata boshi** deyiladi. Koordinata o‘qlari odatda x, y, z yoki Ox, Oy, Oz kabi belgilanadi.

Tekislikda koordinatalar sistemasi quyidagicha kiritiladi.

Ikkita Ox, Oy koordinata o‘qlari shunday kesishsinki,

$(P1)$ ular perpendikulyar bo‘lsin;

$(P2)$ ularning koordinata boshlari ustma-ust tushsin;

$(P3)$ Ox va Oy o‘qlarning musbat yo‘nalishlari ustma-ust tushguncha Ox o‘qini umumiy koordinata boshiga nisbatan eng qisqa burish soat strelkasi harakatiga qarama-qarshi bo‘lsin.

Koordinata boshi, musbat yo‘nalishlari va birlik kesmalari tayinlangan, $(P1)$, $(P2)$, $(P3)$ shartlarni qanoatlantiruvchi tekislik **to‘g‘ri burchakli koordinatalar tekisligi** deyiladi va odatda Oxy kabi belgilanadi.

Endi fazoda koordinatalar sistemasini kiritamiz. Buning uchun o‘ng qo‘l qoidasini bilish talab etiladi. Kafti gorizontal tutilgan o‘ng qo‘lning bosh barmog‘ini gorizontal kerilgan holatda, ko‘rsatkich barmog‘ini gorizontal yoyilgan holatda, o‘rta barmoq kaftga perpendikulyar tutilgan bo‘lsin. Musbat yo‘nalishlarni bosh barmoq, ko‘rsatkich barmoq va o‘rta barmoqning shu holati bo‘yicha aniqlash **o‘ng qo‘l qoidasi** deyiladi.

Uchta Ox, Oy va Oz koordinata o‘qlari shunday kesishsinki,

$(S1)$ ular o‘zaro perpendikulyar bo‘lsin;

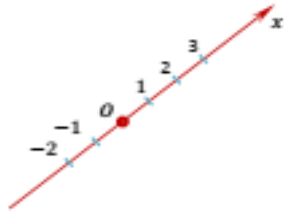
$(S2)$ ularning koordinata boshlari ustma-ust tushsin;

$(S3)$ Ox, Oy va Oz o‘qlarning musbat yo‘nalishlari o‘ng qo‘l qoidasi asosida belgilansin. Bunda Ox o‘qning musbat yo‘nalishi bosh barmoq yo‘nalishi, Oy o‘qning musbat yo‘nalishi ko‘rsatkich barmoq yo‘nalishi

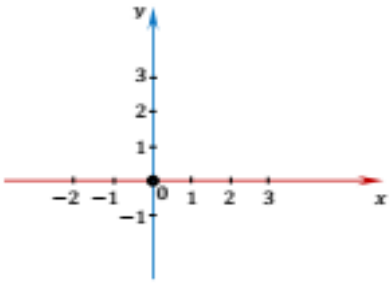
va Oz o‘qning musbat yo‘nalishi o‘rta barmoq yo‘nalishi bilan bir xil bo‘lishi talab etiladi.

Koordinata boshi, musbat yo‘nalishlari va birlik kesmalari tayinlangan, $(S1)$, $(S2)$, $(S3)$ shartlarni qanoatlantiruvchi fazo **to‘g‘ri burchakli koordinatalar fazosi** deyiladi va odatda $Oxyz$ kabi belgilanadi.

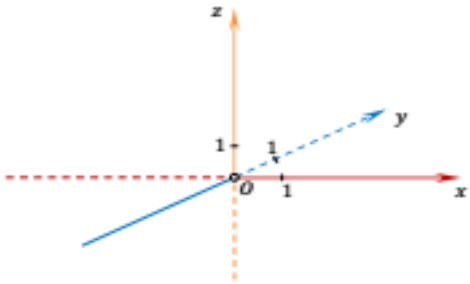
$(P3)$ shart tekislikda, $(S3)$ shart fazoda **orientatsiyani** aniqlaydi deyishadi.



Son o‘qi
(hisob boshi, musbat yo‘nalish va birlik kesma ko‘rsatilgan to‘g‘ri chiziq)



Tekislikdagi Dekart koordinatalar sistemasi



Fazodagi Dekart koordinatalar sistemasi

Demak, **koordinata o‘qi** (yoki **son o‘qi**) deganda koordinata boshi, birlik kesmasi va musbat yo‘nalishi aniqlangan to‘g‘ri chiziq tushuniladi.

Koordinata boshi, koordinata o‘qlari va orientatsiyasi yuqoridagi usulda aniqlangan koordinatalar sistemasi mos ravishda tekislikdagi yoki fazodagi **Dekart koordinatalar sistemasi** deyiladi.

1. Son o‘qidagi (umuman olganda har qanday to‘g‘ri chiziqdagi) har bir nuqtasi bitta va faqat bitta haqiqiy son bilan ifodalanadi. Bu son shu nuqtaning **koordinatasi** deyiladi.

Aytaylik, x son o‘qida A nuqta berilgan bo‘lsin. A nuqtaga mos sonni x_A orqali belgilaylik. U holda x_A son A nuqtaning koordinatasi bo‘ladi. Bu holat $A(x_A)$ kabi belgilanadi.

Son o‘qidagi $A(x_A)$ va $B(x_B)$ nuqtalar orasidagi masofa

$$\rho(A,B) = |x_B - x_A|$$

kabi aniqlanadi. Bu masofa AB orqali ham ifodalanadi: $AB = |x_B - x_A|$.

2. Oxy Dekart koordinatalari sistemasi kiritilgan tekislik **koordinatalar tekisligi** deb ham yuritiladi. Oxy koordinatalar tekisligidagi har bir nuqta ikkita haqiqiy sonlarning tartiblangan jufti orqali yagona usulda (yoki, boshqacha aytganda, o‘zaro bir qiymatli) ifodalanadi. Bu sonlarning tartiblangan jufti mazkur nuqtaning **koordinatalari** deyiladi.

Oxy koordinatalar tekisligidagi A nuqta haqiqiy sonlarning tartiblangan (x_A, y_A) jufti orqali ifodalangan bo‘lsin. Bu holda (x_A, y_A) juftlik A nuqtaning koordinatalari bo‘ladi va $A(x_A, y_A)$ kabi belgilanadi. x_A son A nuqtaning **abssissasi**, y_A son esa A nuqtaning **ordinatasi** deyiladi.

Oxy koordinatalar tekisligida $A(x_A, y_A)$ va $B(x_B, y_B)$ nuqtalar orasidagi masofa

$$AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \qquad (1.1)$$

formula orqali topiladi.

3. $Oxyz$ Dekart koordinatalari sistemasi kiritilgan fazodagi har bir nuqta uchta haqiqiy sonlarning tartiblangan uchligi orqali yagona usulda ifodalanadi. Bu sonlar uchligi mazkur nuqtaning **koordinatalari** deyiladi.

$Oxyz$ koordinatalar tekisligidagi A nuqta haqiqiy sonlarning tartiblangan (x_A, y_A, z_A) uchligi orqali ifodalangan bo‘lsin. Bu holda tartiblangan (x_A, y_A, z_A) uchlik A nuqtaning koordinatalari bo‘ladi va $A(x_A, y_A, z_A)$ kabi belgilanadi. x_A son A nuqtaning **abssissasi**, y_A son A nuqtaning **ordinatasi**, z_A son esa A nuqtaning **applikatasi** deyiladi.

$Oxyz$ koordinatalar tekisligida $A(x_A, y_A, z_A)$ va $B(x_B, y_B, z_B)$ nuqtalar orasidagi masofa

$$AB = \rho(A, B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \qquad (1.2)$$

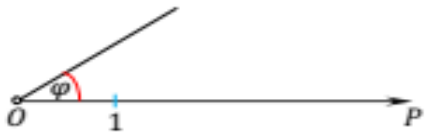
formula orqali topiladi.

4.2. Qutb koordinatalar sistemasi

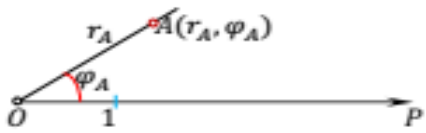
Tekislikda biror nuqtani (kuzatuv nuqtasini) tayinlaymiz. Bu tayinlangan nuqtani O harfi orqali belgilaylik. O nuqtadan chiquvchi biror nurni tayinlab olamiz. Aytaylik, kuzatuvchiga nisbatan o‘ngga yo‘nalgan gorizontal nur tayinlangan bo‘lsin. Bu nurni OP orqali belgilaymiz. OP nurda birlik kesma ajratamiz. OP nurdan boshlab qo‘yiladigan φ burchaklarni qaraymiz. φ burchakning OP nurdan soat strelkasi harakatiga qarama-qarshi yo‘nalishda burilishi **musbat burilish** deb qabul qilinadi. Bunda φ burchakning $[0,\ 2\pi)$ oraliqdagi qiymatlari qaraladi (boshqacha aytganda, k ning har qanday butun qiymatida $\varphi_0 + 2\pi k$ burchak φ_0 burchak bilan ayniylashtiriladi, ya’ni

$$(\star) \qquad \varphi_0 + 2\pi k \cong \varphi_0, \qquad \forall k \in \mathbb{Z}).$$

Yuqorida aniqlangan O nuqta, birlik kesmasi ko‘rsatilgan OP nur va musbat burilishi aniqlangan φ burchakdan tashkil topgan sistema (tekislikdagi) **qutb koordinatalari sistemasi** deyiladi. Bunda O nuqta **qutb**, OP o‘q **qutb o‘qi**, φ burchak **qutb burchagi** deyiladi.

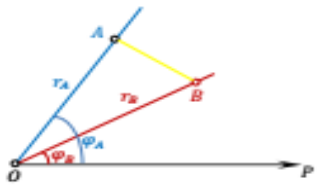


Qutb koordinatalari sistemasi



A nuqtaning qutb koordinatalari

$(\star)$ kelishuvni e’tiborga olganda, tekislikdagi har bir nuqta o‘zining qutb koordinatalari orqali yagona usulda aniqlanadi. Aytaylik, tekislikdagi biror A nuqtaning qutb koordinatalari $(r_A,\ \varphi_A)$, ya’ni $A(r_A,\ \varphi_A)$ bo‘lsin. Bu erda r_A – qutb (O nuqta) dan A nuqtagacha masofa, φ_A – qutb o‘qi (OP o‘q) va OA o‘q orasidagi burchak. r_A masofani A nuqtaning **qutb radiusi**, φ_A burchakni A nuqtaning **qutb burchagi** deyishadi. Bayon



qilingan qutb koordinatlar sistemasi $\{O, r, \varphi\}$ kabi belgilanadi.

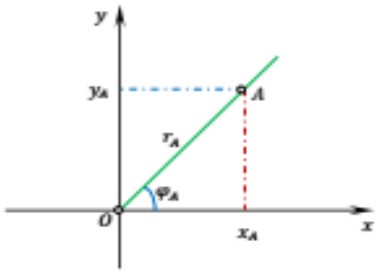
Qutb koordinatalari bilan berilgan ikkita $A(r_A,\ \varphi_A)$ va $B(r_B,\ \varphi_B)$ nuqtalar orasidagi masofa

$$\rho(A,B) = AB = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\varphi_B - \varphi_A)} \qquad (2.1)$$

formula orqali hisoblanadi.

Tekislikdagi A nuqtaning (x_A, y_A) Dekart koordinatalaridan (r_A, φ_A) qutb koordinatalariga

$$r_A = \sqrt{x_A^2 + y_A^2}, \qquad \varphi_A = arctg\frac{y_A}{x_A} \qquad (2.2)$$



tengliklar orqali o‘tiladi.

Teskari o‘tish, ya’ni tekislikdagi A nuqtaning (r_A, φ_A) qutb koordinatalaridan (x_A, y_A) Dekart koordinatalariga o‘tish

$$x_A = r_A \cos\varphi_A, \qquad y_A = r_A \sin\varphi_A \qquad (2.3)$$

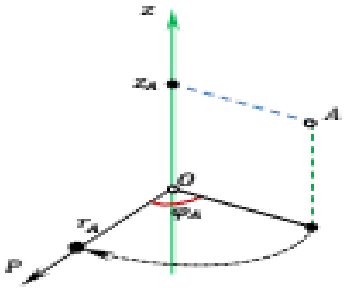
tengliklar orqali amalga oshiriladi.

(2.2) va (2.3) formulalar muhim ahamiyatga ega. Ular o‘zgaruvchilarni almashtirishning yaqqol misoli bo‘ladi va yoy uzunligini, egri chiziqli trapetsiya yuzasini, aylanish jismlarining sirti yuzasini hamda hajmini hisoblashda ishlatiladi.

Mustaqil ishlash uchun topshiriqlar.

(1.1) formuladan (2.1) formulani keltirib chiqaring.

(2.1) formuladan (1.1) formulani keltirib chiqaring.

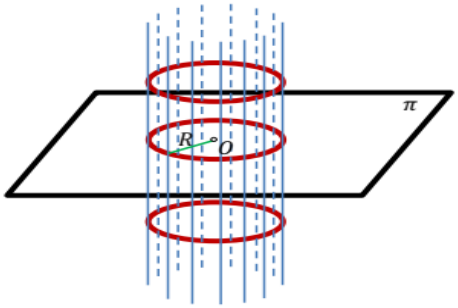


4.3. Silindrik koordinatalar sistemasi

Qutb koordinatalari sistemasi tekislikdagi nuqtaning vaziyatini aniqlaydi. Fazodagi nuqtaning vaziyatini aniqlash uchun silindrik koordinatalar sistemasini kiritamiz.

Avvalo silindr va silindrik chiziq tushunchasini kiritamiz.

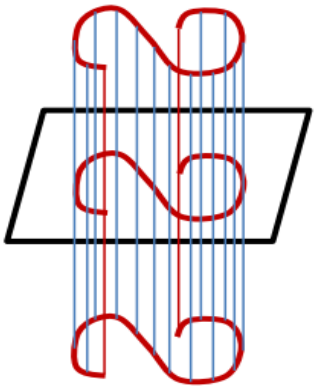
Ma’lumki, tekislikning berilgan nuqtasidan bir xil masofada yotgan nuqtalari to‘plami **aylana** deyiladi. Berilgan nuqta aylananing **markazi**, markazdan aylananing biror (yoki ixtiyoriy) nuqtasigacha bo‘lgan masofa aylananing **radiusi** deyiladi. SHu tekislikni aylana nuqtalari orqali kesib o‘tuvchi parallel to‘g‘ri chiziqlar to‘plami **silindr** deyiladi. Bunda har bir to‘g‘ri chiziq silindrning **yasovchisi** deyiladi. Agar silindrning yasovchisi aylana tekisligiga perpendikulyar bo‘lsa, bu silindr **to‘g‘ri silindr** deyiladi, aks holda **og‘ma silindr** deyiladi.



To`g`ri silindr

π tekislikdagi
γ egrichiziq

Tekislikdagi ixtiyoriy γ egri chiziqni qaraylik (γ – “gamma” deb o‘qiladi, grek harfi). Shu tekislikni γ egri chiziq nuqtalari orqali kesib o‘tuvchi parallel to‘g‘ri chiziqlar to‘plami **silindrik sirt** deyiladi. Bunda har bir to‘g‘ri chiziq silindrik sirtning **yasovchisi** deyiladi. Agar silindrik sirtning yasovchisi γ egri chiziq tekisligiga perpendikulyar bo‘lsa, bu silindrik sirt **to‘g‘ri silindrik silindrik** deyiladi, aks holda **og‘ma silindrik sirt** deyiladi.


Silindrik sirt

Endi fazoda $\{O, r, \varphi\}$ qutb koordinatalar sistemasi kiritilgan tekislik va uni O qutb orqali perpendikulyar kesib o‘tuvchi Oz o‘q berilgan bo‘lsin. Bunda Oz o‘q shunday yo‘nalganki, yo‘nalishning “oxiri”dan kuzatganda qutb tekisligidagi musbat burilish soat strelkasi harakatiga qarama-qarshi bo‘lsin. Hosil bo‘lgan $\{O,\ r,\ \varphi,\ z\}$ sistema **silindrik koordinatalar sistemasi** deyiladi.

Fazodagi har bir nuqta silindrik koordinatalar sistemasida uchta r, φ, z son orqali bir qiymatli ifodalanadi, bu erda $0 \le r < +\infty$, $0 \le \varphi < 2\pi$, $-\infty < z < +\infty$. Bu sonlar mazkur nuqtaning silindrik koordinatalari deyiladi.

Aytaylik, fazodagi biror A nuqtaning silindrik koordinatalari r_A, φ_A, z_A bo‘lsin. Bu holat $A(r_A,\ \varphi_A,\ z_A)$ kabi yoziladi. r_A son Oz o‘qdan A nuqtagacha masofani anglatadi va A nuqtaning **radial masofasi** deyiladi; φ_A sonni A nuqtaning **azimuti** deyishadi; z_A songa nibatan A nuqtaning **balandligi** iborasini ishlatishadi.

Fazodagi A nuqtaning (x_A, y_A, z_A) Dekart koordinatalaridan uning (r_A, φ_A, z_A) silindrik koordinatalariga

$$r_A = \sqrt{x_A^2 + y_A^2}, \qquad \varphi_A = arctg\frac{y_A}{x_A}, \qquad z_A = z_A \tag{3.1}$$

tengliklar orqali o‘tiladi.

Teskari o‘tish, ya’ni fazodagi A nuqtaning (r_A, φ_A, z_A) silindrik koordinatalaridan uning (x_A, y_A, z_A) Dekart koordinatalariga o‘tish

$$x_A = r_A \cos\varphi_A, \qquad y_A = r_A \sin\varphi_A, \qquad z_A = z_A \tag{3.2}$$

tengliklar orqali amalga oshiriladi.

Silindrik koordinatalari bilan berilgan ikkita $A(r_A, \varphi_A, z_A)$ va $B(r_B, \varphi_B, z_B)$ nuqtalar orasidagi masofa

$$\rho(A, B) = AB = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\varphi_B - \varphi_A) + (z_B - z_A)^2} \tag{3.3}$$

formula orqali hisoblanadi.

Mustaqil ishlash uchun topshiriqlar.

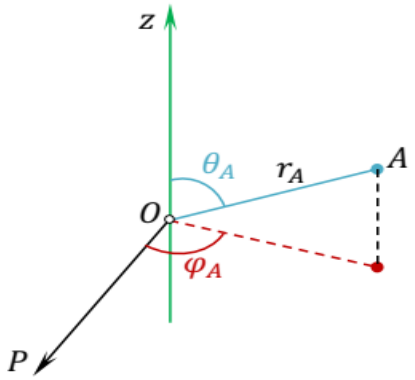
(1.2) formuladan (3.3) formulani keltirib chiqaring.

(3.3) formuladan (1.2) formulani keltirib chiqaring.

4.4. Sferik koordinatalar sistemasi

Fazoning berilgan nuqtasidan bir xil masofada yotgan nuqtalari to‘plami **sfera** deyiladi. Berilgan nuqta sferaning **markazi**, markazdan sferaning biror (yoki ixtiyoriy) nuqtasigacha bo‘lgan masofa sferaning **radiusi** deyiladi.

Fazoda $\{O, r, \varphi\}$ qutb koordinatalar sistemasi kiritilgan tekislik va uni O qutb orqali perpendikulyar kesib o‘tuvchi Oz o‘q berilgan bo‘lsin. Bunda Oz o‘q shunday yo‘nalganki, yo‘nalishning “oxiri”dan kuzatganda qutb tekisligidagi musbat burilish soat strelkasi harakatiga qarama-qarshi bo‘lsin. Fazodagi har bir A nuqtaning O nuqtaga nisbatan vaziyati undan O nuqtagacha bo‘lgan r_A masofa, uning qutb koordinatalar tekisligiga proeksiyasi bilan qutb o‘qi orasidagi φ_A burchak, OA kesma bilan Oz o‘qining musbat yo‘nalishi orasidagi θ_A burchak orqali aniqlash mumkin. Bu holat $A(r_A, \varphi_A, \theta_A)$ kabi yoziladi. Hosil bo‘lgan $\{O, r, \varphi, \theta\}$ sistema **sferik koordinatalar sistemasi** deyiladi.



Qutb koordinatalar tekisligini sferik koordinatalar sistemasining **fundamental tekisligi** deb ham yuritishadi; r masofani **radius**, φ burchakni **azimut**, Oz o‘qini **zenit**, θ burchakni **zenit burchagi** deyishadi.

$\{O, r, \varphi, \theta\}$ sferik koordinatalar sistemasidagi kattaliklar

$$0 \le r < +\infty,$$
$$0 \le \varphi < 2\pi,$$
$$0 \le \theta \le \pi$$

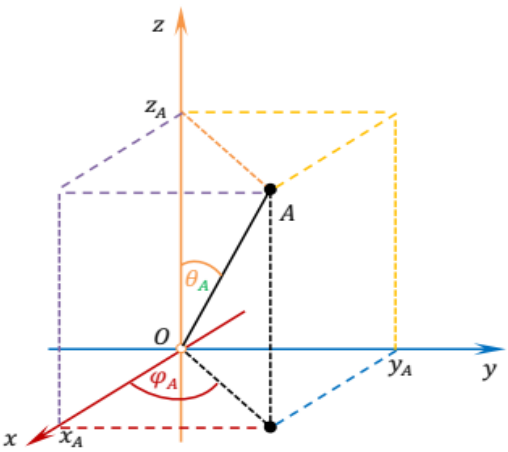
shartlarni qanoatlantirishi talab etiladi.

A nuqtaning sferik $(r_A, \varphi_A, \theta_A)$ koordinatalaridan Dekart (x_A, y_A, z_A) koordinatalariga quyidagicha o‘tiladi:

$$x_A = r_A \cos\varphi_A \sin\theta_A,$$
$$y_A = r_A \sin\varphi_A \sin\theta_A,$$
$$z_A = r_A \cos\theta_A.$$

Teskari, ya’ni A nuqtaning (x_A, y_A, z_A) Dekart koordinatalaridan uning sferik $(r_A, \varphi_A, \theta_A)$ koordinatalariga o‘tish quyidagi almashtirishlar orqali bajariladi:

$$r_A = \sqrt{x_A^2 + y_A^2 + z_A^2},$$
$$\varphi_A = arctg\frac{y_A}{x_A},$$
$$\theta_A = arctg\frac{\sqrt{x_A^2 + y_A^2}}{z_A}.$$



Mustahkamlash uchun mashqlar
Koordinatalar sistemalariga oid masalalar

1. Quyidagi nuqtalar berilgan. Bu nuqtalarning Dekart koordinatalar sistemasidagi geometrik o‘rnini ko‘rsating.

1) $A(0,\ 1)$, 2) $B(3,\ -4)$, 3) $C(-2,-5)$, 4) $D(-4,\ 2)$.

2. Quyidagi nuqtalar berilgan. Bu nuqtalarning qutb koordinatalar sistemasidagi geometrik o‘rnini ko‘rsating.

1) $A(2,\ 30^0)$, 2) $B\left(3,\ \frac{5\pi}{4}\right)$, 3) $C\left(2,\frac{5\pi}{6}\right)$, 4) $D(4,\ \pi)$.

3. Dekart koordinatalari bilan berilgan nuqtalarning qutb koordinatalarini ko‘rsating.

1) $A(0,\ 1)$, 2) $B(3,\ -4)$, 3) $C(-2,-5)$, 4) $D(-4,\ 2)$.

4. Qutb koordinatalari bilan berilgan nuqtalarning Dekart koordinatalarini ko‘rsating.

1) $A(2,\ 30^0)$, 2) $B\left(3,\ \frac{5\pi}{4}\right)$, 3) $C\left(2,\frac{5\pi}{6}\right)$, 4) $D(4,\ \pi)$.

5. Dekart koordinatalari bilan berilgan nuqtalarning silindrik koordinatalarini ko‘rsating.

1) $A(0,\ 1,-3)$, 2) $B(3,\ -4,\ 2)$, 3) $C(-2,-5,\ 1)$, 4) $D(-4,\ 2,-1)$.

6. Dekart koordinatalari bilan berilgan nuqtalarning sferik koordinatalarini ko‘rsating.

1) $A(0,\ 1,-3)$, 2) $B(3,\ -4,\ 2)$, 3) $C(-2,-5,\ 1)$, 4) $D(-4,\ 2,-1)$.

7. Silindrik koordinatalari bilan berilgan nuqtalarning Dekart koordinatalarini ko‘rsating.

1) $A(2,\ 30^0,\ 3)$, 2) $B\left(3,\ \frac{5\pi}{4},-2\right)$, 3) $C\left(2,\frac{5\pi}{6},\ 4\right)$, 4) $D(4,\ \pi,-1)$.

8. Sferik koordinatalari bilan berilgan nuqtalarning Dekart koordinatalarini ko‘rsating.

1) $A(2,\ 30^0,-60^0)$, 2) $B\left(3,\ \frac{5\pi}{4},\ \frac{\pi}{4}\right)$, 3) $C\left(2,\frac{5\pi}{6},\frac{\pi}{6}\right)$, 4) $D(4,\ \pi,\ 0)$.

9. Dekart koordinatalari bilan berilgan nuqtalar orasidagi masofani toping.

1) $A(0,\ 1)$, $B(3,\ -3)$, 2) $C(2,\ 8)$, 4) $D(-10,\ 3)$.

10. Qutb koordinatalari bilan berilgan nuqtalar orasidagi masofani toping.

1) $A\left(1,\ \frac{\pi}{2}\right)$, $B\left(3\sqrt{2},\ \frac{7\pi}{4}\right)$, 2) $C\left(2\sqrt{17},\ arctg\ 4\right)$, $D\left(\sqrt{109},\ \pi - arctg\frac{3}{10}\right)$.

11. Silindrik koordinatalari bilan berilgan nuqtalar orasidagi masofani toping.

$$1)\ A\left(1,\ \frac{\pi}{2},\ 2\right),\ B\left(3\sqrt{2},\ \frac{7\pi}{4},\ 14\right),$$
$$2)\ C\left(2\sqrt{17},\ arctg\ 4,\ 3\right),\ D\left(\sqrt{109},\ \pi - arctg\frac{3}{10},\ 3\right).$$

12. Sferik koordinatalari bilan berilgan nuqtalar orasidagi masofani toping.
1) $A(2,\ 30^0, -60^0)$, $B\left(3,\ \frac{5\pi}{4},\ \frac{\pi}{4}\right)$, 2) $C\left(2, \frac{5\pi}{6}, \frac{\pi}{6}\right)$, $D(4,\ \pi,\ 0)$.

13. Uchlari $A(0,\ 1)$, $B(3,\ -3)$, $C(0, -1)$ Dekart koordinatalari bilan berilgan uchburchak perimetrini toping.

14. Uchlari $A\left(1, \frac{\pi}{2}\right)$, $B\left(3\sqrt{2}\ ,\ \frac{7\pi}{4}\right)$, $C\left(1, \frac{3\pi}{2}\right)$ qutb koordinatalari bilan berilgan uchburchak perimetrini toping.

13. Uchlari $A\left(1,\ \frac{\pi}{2}\right)$, $B\left(3\sqrt{2},\ \frac{7\pi}{4}\right)$, $C\left(2\sqrt{17},\ arctg\ 4\right)$ silindrik koordinatalari bilan berilgan uchburchak perimetrini toping.

15. Uchlari $A(2,\ 30^0, -60^0)$, $B\left(3,\ \frac{5\pi}{4},\ \frac{\pi}{4}\right)$, $C\left(2, \frac{5\pi}{6}, \frac{\pi}{6}\right)$ sferik koordinatalari bilan berilgan uchburchak perimetrini toping.

O`z-o`zini tekshirish uchun savollar

1. Qanday koordinatalar sistemalarini bilasiz?
2. Dekart koordinatalar sistemasi kiritilishini tushuntiring.
3. Qutb koordinatalar sistemasi kiritilishini tushuntiring.
4. Silindrik koordinatalar sistemasi kiritilishini tushuntiring.
5. Sferik koordinatalar sistemasi kiritilishini tushuntiring.

O`z-o`zini tekshirish uchun testlar

Koordinatalar sistemasi oid testlar

1. Son o‘qidagi $A(x_A)$ va $B(x_B)$ nuqtalar orasidagi masofa qaysi formula orqali topiladi.
a) $\rho(A,B) = |x_B - x_A|$ b) $\rho(A,B) = |x_B + x_A|$
c) $AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$
d) $AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 - (y_B - y_A)^2}$
2. Oxy koordinatalar tekisligida $A(x_A, y_A)$ va $B(x_B, y_B)$ nuqtalar orasidagi masofa qaysi formula orqali topiladi?
a) $\rho(A,B) = |x_B - x_A|$ b) $\rho(A,B) = |x_B + x_A|$
c) $AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$
d) $AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 - (y_B - y_A)^2}$

3. $Oxyz$ koordinatalar tekisligida $A(x_A, y_A, z_A)$ va $B(x_B, y_B, z_B)$ nuqtalar orasidagi masofa qaysi formula orqali topiladi?

a) $\rho(A,B) = |x_B - x_A|$ b) $\rho(A,B) = |x_B + x_A|$

c) $AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

d) $AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$

4. Qutb koordinatalari bilan berilgan ikkita $A(r_A,\ \varphi_A)$ va $B(r_B,\ \varphi_B)$ nuqtalar orasidagi masofa qaysi formula orqali topiladi?

a) $\rho(A,B) = AB = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\varphi_B - \varphi_A)}$

b) $\rho(A,B) = |x_B - x_A|$

c) $AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

d) $AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$

5. Silindrik koordinatalari bilan berilgan ikkita $A(r_A, \varphi_A, z_A)$ va $B(r_B, \varphi_B, z_B)$ nuqtalar orasidagi masofa qaysi formula orqali topiladi?

a) $\rho(A,B) = AB = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\varphi_B - \varphi_A)}$

b) $\rho(A,B) = AB = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\varphi_B - \varphi_A) + (z_B - z_A)^2}$

c) $AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

d) $AB = \rho(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$

6. Qutb koordinatlar sistemasi qanday belgilanadi?

a) $\{O, r, \varphi\}$ b) $Oxyz$ c) Oxy d) $\{O,\ r,\ \varphi,\ z\}$

7. Silindrik koordinatlar sistemasi qanday belgilanadi?

a) $\{O, r, \varphi\}$ b) $Oxyz$ c) Oxy d) $\{O,\ r,\ \varphi,\ z\}$

8. Sferik koordinatlar sistemasi qanday belgilanadi?

a) $\{O, r, \varphi\}$ b) $Oxyz$ c) $\{O, r, \varphi, \theta\}$ d) $\{O,\ r,\ \varphi,\ z\}$

9. Tekislikdagi A nuqtaning (x_A, y_A) Dekart koordinatalaridan (r_A, φ_A) qutb koordinatalariga qaysi formula orqali o‘tiladi?

a) $r_A = \sqrt{x_A^2 + y_A^2}$, $\varphi_A = arctg\frac{y_A}{x_A}$

b) $x_A = r_A \cos\varphi_A$, $y_A = r_A \sin\varphi_A$

c) $r_A = \sqrt{x_A^2 + y_A^2}$, $\varphi_A = arctg\frac{y_A}{x_A}$, $z_A = z_A$

d) $\rho(A,B) = AB = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\varphi_B - \varphi_A) + (z_B - z_A)^2}$

10. Tekislikdagi A nuqtaning (r_A, φ_A) qutb koordinatalaridan (x_A, y_A) Dekart koordinatalariga qaysi formula orqali o‘tiladi?

a) $r_A = \sqrt{x_A^2 + y_A^2}$, $\varphi_A = arctg\frac{y_A}{x_A}$

b) $x_A = r_A \cos\varphi_A$, $y_A = r_A \sin\varphi_A$

c) $r_A = \sqrt{x_A^2 + y_A^2}$, $\varphi_A = arctg\frac{y_A}{x_A}$, $z_A = z_A$

d)) $\rho(A,B) = AB = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\varphi_B - \varphi_A) + (z_B - z_A)^2}$

11. Fazodagi A nuqtaning (x_A, y_A, z_A) Dekart koordinatalaridan uning (r_A, φ_A, z_A) silindrik koordinatalariga qaysi formula orqali o‘tiladi?

a) $r_A = \sqrt{x_A^2 + y_A^2}$, $\varphi_A = arctg\frac{y_A}{x_A}$

b) $x_A = r_A \cos\varphi_A$, $\quad y_A = r_A \sin\varphi_A$
c) $r_A = \sqrt{x_A^2 + y_A^2}$, $\quad \varphi_A = arctg\frac{y_A}{x_A}$, $\quad z_A = z_A$
d) $\rho(A,B) = AB = \sqrt{r_A^2 + r_B^2 - 2r_A r_B \cos(\varphi_B - \varphi_A) + (z_B - z_A)^2}$
12. Fazodagi A nuqtaning (r_A, φ_A, z_A) silindrik koordinatalaridan uning (x_A, y_A, z_A) Dekart koordinatalariga qaysi formula orqali o‘tiladi?
a) $r_A = \sqrt{x_A^2 + y_A^2}$, $\quad \varphi_A = arctg\frac{y_A}{x_A}$
b) $x_A = r_A \cos\varphi_A$, $\quad y_A = r_A \sin\varphi_A$
c) $r_A = \sqrt{x_A^2 + y_A^2}$, $\quad \varphi_A = arctg\frac{y_A}{x_A}$, $\quad z_A = z_A$
d) $x_A = r_A \cos\varphi_A$, $\quad y_A = r_A \sin\varphi_A$, $\quad z_A = z_A$
13. A nuqtaning sferik $(r_A, \varphi_A, \theta_A)$ koordinatalaridan Dekart (x_A, y_A, z_A) koordinatalariga qaysi formula orqali o‘tiladi?
a) $x_A = r_A \cos\varphi_A \sin\theta_A$, $y_A = r_A \sin\varphi_A \sin\theta_A$, $z_A = r_A \cos\theta_A$.
b) $x_A = r_A \cos\varphi_A$, $\quad y_A = r_A \sin\varphi_A$
c) $r_A = \sqrt{x_A^2 + y_A^2}$, $\quad \varphi_A = arctg\frac{y_A}{x_A}$, $\quad z_A = z_A$
d) $x_A = r_A \cos\varphi_A$, $\quad y_A = r_A \sin\varphi_A$, $\quad z_A = z_A$
14. A nuqtaning (x_A, y_A, z_A) Dekart koordinatalaridan uning sferik $(r_A, \varphi_A, \theta_A)$ koordinatalariga qaysi formula orqali o‘tiladi?
a) $r_A = \sqrt{x_A^2 + y_A^2 + z_A^2}$, $\varphi_A = arctg\frac{y_A}{x_A}$, $\theta_A = arctg\frac{\sqrt{x_A^2+y_A^2}}{z_A}$.
b) $x_A = r_A \cos\varphi_A$, $\quad y_A = r_A \sin\varphi_A$
c) $r_A = \sqrt{x_A^2 + y_A^2}$, $\quad \varphi_A = arctg\frac{y_A}{x_A}$, $\quad z_A = z_A$
d) $x_A = r_A \cos\varphi_A$, $\quad y_A = r_A \sin\varphi_A$, $\quad z_A = z_A$
15. Son o‘qidagi $A(-2)$ va $B(3)$ nuqtalar orasidagi masofani toping.
a) $\rho(A,B) = 5$ $\quad$ b) $\rho(A,B) = 1$ $\quad$ c) $\rho(A,B) = 3$ $\quad$ d) $\rho(A,B) = 2$
16. Oxy koordinatalar tekisligida $A(0,3)$ va $B(4,0)$ nuqtalar orasidagi masofani toping.
a) $\rho(A,B) = 5$ $\quad$ b) $\rho(A,B) = 1$ $\quad$ c) $\rho(A,B) = 3$ $\quad$ d) $\rho(A,B) = 2$
17. $Oxyz$ koordinatalar fazosidagi $A(0,4,0)$ va $B(3,0,12)$ nuqtalar orasidagi masofani toping.
a) $\rho(A,B) = 15$ $\quad$ b) $\rho(A,B) = 10$ $\quad$ c) $\rho(A,B) = 13$ $\quad$ d) $\rho(A,B) = 12$
18. Qutb koordinatalari bilan berilgan ikkita $A\left(5, \frac{\pi}{4}\right)$ va $B\left(9, \frac{\pi}{4}\right)$ nuqtalar orasidagi masofani toping.
a) $\rho(A,B) = 5$ $\quad$ b) $\rho(A,B) = 1$ $\quad$ c) $\rho(A,B) = 3$ $\quad$ d) $\rho(A,B) = 4$
19. Silindrik koordinatalari bilan berilgan ikkita $A\left(4, \frac{\pi}{4}, 1\right)$ va $B\left(13, \frac{\pi}{4}, 13\right)$ nuqtalar orasidagi orasidagi masofani toping.
a) $\rho(A,B) = 15$ $\quad$ b) $\rho(A,B) = 10$ $\quad$ c) $\rho(A,B) = 13$ $\quad$ d) $\rho(A,B) = 12$
20. Tekislikdagi A nuqtaning $(3,3)$ Dekart koordinatalaridan (r_A, φ_A) qutb koordinatalariga o‘ting?
a) $r_A = 3\sqrt{2}$, $\quad \varphi_A = \frac{\pi}{4}$

b) $r_A = 3\sqrt{2}$, $\varphi_A = \frac{\pi}{3}$
c) $r_A = 3$, $\varphi_A = \frac{\pi}{4}$
d) $r_A = 3\sqrt{2}$, $\varphi_A = \frac{\pi}{2}$

21. Tekislikdagi A nuqtaning $\left(3\sqrt{2}, \frac{\pi}{4}\right)$ qutb koordinatalaridan (x_A, y_A) Dekart koordinatalariga o‘ting.
a) $A(3,3)$ b) $A(3,0)$ c) $A(0,3)$ d) $A(3,4)$

22. Fazodagi A nuqtaning $(3,3,3)$ Dekart koordinatalaridan uning (r_A, φ_A, z_A) silindrik koordinatalariga o‘ting.
a) $r_A = 3\sqrt{2}$, $\varphi_A = \frac{\pi}{4}$, $z_A = 3$ b) $r_A = 3$, $\varphi_A = \frac{\pi}{4}$, $z_A = 2$
c) $r_A = 3\sqrt{2}$, $\varphi_A = \frac{\pi}{4}$, $z_A = 0$ d) $r_A = 3$, $\varphi_A = \frac{\pi}{4}$, $z_A = 1$

23. Fazodagi A nuqtaning $\left(3\sqrt{2}, \frac{\pi}{4}, 3\right)$ silindrik koordinatalaridan uning (x_A, y_A, z_A) Dekart koordinatalariga o‘ting.
a) $A(3,3,3)$ b) $A(3,0,3)$ c) $A(3,1,3)$ d) $A(3,2,3)$

24. A nuqtaning $(1,1,1)$ Dekart koordinatalaridan uning sferik $(r_A, \varphi_A, \theta_A)$ koordinatalariga o‘ting.
a) $r_A = \sqrt{3}$, $\varphi_A = \frac{\pi}{4}$, $\theta_A = arctg\sqrt{2}$. b) $r_A = \sqrt{2}$, $\varphi_A = \frac{\pi}{4}$, $\theta_A = arctg\sqrt{2}$.
c) $r_A = \sqrt{5}$, $\varphi_A = \frac{\pi}{4}$, $\theta_A = arctg\sqrt{2}$. d$r_A = \sqrt{7}$, $\varphi_A = \frac{\pi}{4}$, $\theta_A = arctg\sqrt{2}$.

25. A nuqtaning sferik $\left(\sqrt{3}, \frac{\pi}{4}, arctg\sqrt{2}\right)$ koordinatalaridan Dekart (x_A, y_A, z_A) koordinatalariga o‘ting.
a) $A(1,1,1)$ b) $A(1,2,1)$ c) $A(1,2,1)$ d) $A(1,2,3)$

26. Son o‘qidagi $A(-6)$ va $B(7)$ nuqtalar orasidagi masofani toping.
a) $\rho(A,B) = 15$ b) $\rho(A,B) = 13$ c) $\rho(A,B) = 3$ d) $\rho(A,B) = 12$

27. Oxy koordinatalar tekisligida $A(0,12)$ va $B(5,0)$ nuqtalar orasidagi masofani toping.
a) $\rho(A,B) = 5$ b) $\rho(A,B) = 12$ c) $\rho(A,B) = 13$ d) $\rho(A,B) = 20$

28. $Oxyz$ koordinatalar fazosidagi $A(4,0,0)$ va $B(0,3,12)$ nuqtalar orasidagi masofani toping.
a) $\rho(A,B) = 15$ b) $\rho(A,B) = 10$ c) $\rho(A,B) = 13$ d) $\rho(A,B) = 12$

29. Qutb koordinatalari bilan berilgan ikkita $A\left(1, \frac{\pi}{4}\right)$ va $B\left(5, \frac{\pi}{4}\right)$ nuqtalar orasidagi masofani toping.
a) $\rho(A,B) = 5$ b) $\rho(A,B) = 1$ c) $\rho(A,B) = 3$ d) $\rho(A,B) = 4$

30. Tekislikdagi A nuqtaning $(4,4)$ Dekart koordinatalaridan (r_A, φ_A) qutb koordinatalariga o‘ting?
a) $r_A = 4\sqrt{2}$, $\varphi_A = \frac{\pi}{4}$ b) $r_A = 3\sqrt{2}$, $\varphi_A = \frac{\pi}{3}$
c) $r_A = 4$, $\varphi_A = \frac{\pi}{4}$ d) $r_A = 3\sqrt{2}$, $\varphi_A = \frac{\pi}{2}$

5-§. VEKTORLAR

5. Vektorlar

Vektorlar kundalik hayotimizning ajralmas qismi hisoblanadi. Har kuni suhbatlarimizda ishlatiladigan va biz ko‘nikib ketgan ushbu iboralarga e’tibor bering:

(Y) “sharq tomonga harakatlanish”, “g‘arb tomonga harakatlanish”, “janub tomonga harakatlanish”, “shimol tomonga harakatlanish”, “tepaga harakatlanish”, “pastga harakatlanish”, “o‘ngga harakatlanish”, “chapga harakatlanish”, “oldinga harakatlanish”, “orqaga harakatlanish”, “burchak bo‘ylab harakatlanish” va hakozo. Bular **yo‘nalishni** anglatadi;

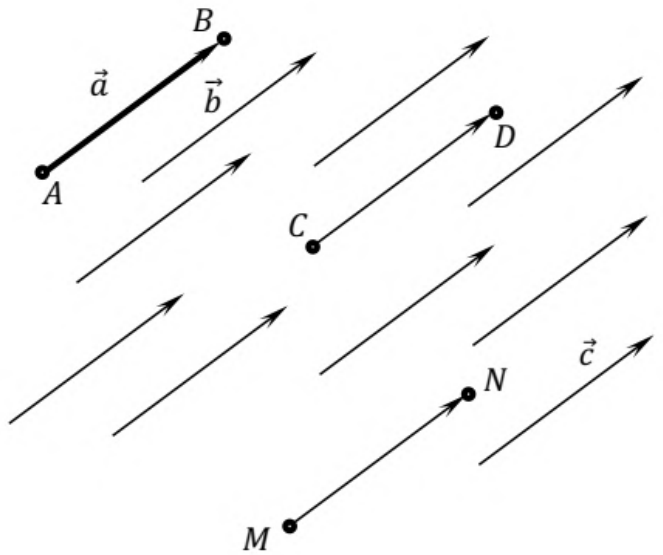
(M) “90 km/soat tezlik bilan harakatlandi”, “500 m ga siljidi”, “Yerning erkin tushish tezlanishi 9,8 m/s^2” va hakozo. Bular **miqdorni ko‘rsatadi**.

(V) “90 km/soat tezlik bilan sharq tomonga harakatlandi”, “Yerga 9,8 m/s^2 tezlanish bilan tushadi”, “o‘ng tomonga 500 m siljidi” va hakozo. Bular **vektor kattaliklar** hisoblanadi.

Demak, **vektor** ham **miqdor**ga, ham **yo‘nalish**ga ega ekan.

5.1. Vektor tushunchasi

Uzunliklari teng, yo‘nalishlari bir xil bo‘lgan barcha kesmalar to‘plami **vektor** deyiladi. Vektorlar lotin alifbosining kichik harflari bilan ustiga strelka qo‘yilgan holda $\vec{a}$, $\vec{b}$, $\vec{c}$, …, yoki boshi va oxiri bilan berilgan vektorlar $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{MN}$, ... kabi belgilanadi.



Rasmga diqqat qiling. Nechta vektor berilgan? Rasmda berilgan kesmalar ko‘p bo‘lsa ham, ularning uzunliklari va yo‘nalishlari bir xil bo‘lgani uchun rasmda bitta vektor berilgan:

$$\vec{a} = \vec{b} = \vec{c} = \overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{MN}.$$

Diqqat! Vektor berilgan bo‘lishi shartlarini eslab qoling:

Vektor berilgan bo‘lishi uchun uning

1) UZUNLIGI va

2) YO‘NALISHI

berilgan bo‘lishi ham zaruriy, ham yetarli shart hisoblanadi.

Ko‘pchilik hollarda “vektor berilgan” iborasi o‘rnigi unga ekvivalent “vektor aniqlangan” iborasi ishlatiladi.

Dekart koordinatlari sistemasi kiritilgan tekislikdagi yoki fazodagi har qanday vektor mos ravishda ikkita yoki uchta son bilan aniqlanadi:

$$\vec{a} = (a_1, a_2) \text{ (tekislikda)} \qquad \text{yoki} \quad \vec{a} = (a_1, a_2, a_3) \text{ (fazoda).}$$

Mazkur ikkita a_1, a_2 yoki uchta a_1, a_2, a_3 sonlar $\vec{a}$ vektorning mos ravishda tekislikdagi yoki fazodagi koordinatalari deyiladi. Bu holda vektor koordinatali aniqlangan deyiladi.

Vektor koordinatali berilgan deyishimiz uchun uning koordinatalari ma'lum bo'lishi zarur va etarli.

Bu vektorlarning uzunliklari

$$\vec{a} = \sqrt{a_1^2 + a_2^2} \text{ (tekislikda)} \qquad \text{yoki} \quad \vec{a} = \sqrt{a_1^2 + a_2^2 + a_3^2} \text{ (fazoda).}$$

formulalar orqali topiladi.

Bu vektorlarning yo'nalishlari koordinata boshi $O(0,0)$ (yoki $O(0,0,0)$) dan $A(a_1, a_2)$ (yoki mos ravishda $A(a_1, a_2, a_3)$) nuqta tomonga yo'nalish orqali aniqlanadi. Boqacha aytganda,

$$\vec{a} = \overrightarrow{OA}.$$

Boshi $A(a_1,\ a_2,\ a_3)$ nuqtada, oxiri $B(b_1,\ b_2,\ b_3)$ nuqtada bo'lgan $\vec{a}$ vektorning koordinatalari quyidagicha topiladi

$$\vec{a} = \overrightarrow{AB} = (b_1 - a_1,\ b_2 - a_2,\ b_3 - a_3).$$

Boshi va oxiri ustma-ust tushadigan vektor nol-vektor deyiladi va $\vec{0}$ kabi belgilanadi. Nol-vektorning uzunligi nolga teng. Nol-vektorning yo'nalishi aniqlanmagan, yoki yo'nalishga ega emas. Nol-vektorning koordinatalari nollardan tashkil topgan bo'ladi: $\vec{0} = (0,\ 0,\ 0)$.

5.2. Vektorlar ustida chiziqli amallar

$\vec{a}$ vektorning λ **songa ko'paytmasi** deb,

1) uzunligi $|\lambda| \cdot |\vec{a}|$ hamda
2) yo'nalishi

$\lambda > 0$ bo'lganda $\vec{a}$ vektorning yo'nalishi bilan bir xil;

$\lambda < 0$ bo'lganda $\vec{a}$ vektorning yo'nalishiga qarama-qarshi

bo'lgan $\lambda\vec{a}$ **vektorga** aytiladi.

$\vec{a} = (a_1, a_2, a_3)$ vektorning λ **songa ko'paytmasi**

$$\lambda\vec{a} = (\lambda a_1, \lambda a_2, \lambda a_3)$$

vektor bo'ladi.

Mustaqil ishlash uchun topshiriq. $\vec{a}$ vektorning λ songa ko'paytirishda ushbu $\lambda > 1$, $0 < \lambda < 1$, $\lambda < 0$ holatlarni aks ettiruvchi chizmalar chizing.

Parallel to'g'ri chiziqlarda yotuvchi vektorlar kollinear vektorlar deyiladi. $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo'lsa, $\vec{a} \mid\mid \vec{b}$ belgilash ishlatiladi. Kollinear $\vec{a}$ va $\vec{b}$ vektorlar bir xil yo'nalishga ega bo'lsa, $\vec{a} \uparrow\uparrow \vec{b}$ belgilash, qarama-qarshi yo'nalishlarga ega bo'lsa, $\vec{a} \uparrow\downarrow \vec{b}$ belgilash ishlatiladi.

$\vec{a} = (a_1, a_2, a_3)$ va $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorlarning **kollinearlik sharti**:

$$\vec{a} \mid\mid \vec{b} \Leftrightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}.$$

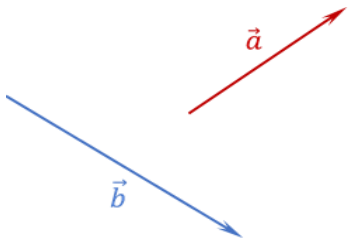
Mustaqil ishlash uchun topshiriqlar.

1) $\vec{a} \uparrow\uparrow \vec{b}$ bo‘lish shartini yozing.

2) $\vec{a} \uparrow\downarrow \vec{b}$ bo‘lish shartini yozing.

$\vec{a} = (a_1, a_2, a_3)$ va $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorlarning **teng** bo‘lish sharti:

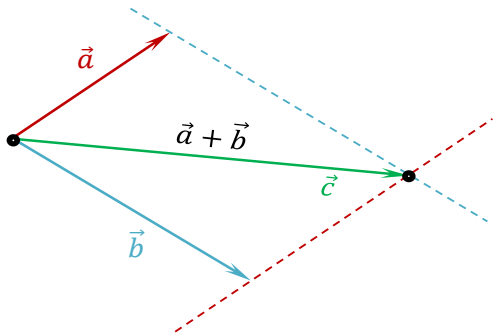
$$\vec{a} = \vec{b} \Leftrightarrow \begin{cases} a_1 = b_1, \\ a_2 = b_2, \\ a_3 = b_3. \end{cases}$$



Endi vektorlar **yig‘indisini** aniqlaymiz. Buning uchun berilgan $\vec{a}$ va $\vec{b}$ vektorlar ustida quyidagi amallarni bajaramiz:

(1) Bu vektorlarning boshlarini umumiy nuqtaga keltiramiz;

(2) $\vec{a}$ vektorning oxiridan $\vec{b}$ vektorga parallel to‘g‘ri chiziq o‘tkazamiz;



(3) $\vec{b}$ vektorning oxiridan $\vec{a}$ vektorga parallel to‘g‘ri chiziq o‘tkazamiz;

(4) $\vec{a}$ va $\vec{b}$ vektorlarning umumiy boshidan o‘tkazilgan to‘g‘ri chiziqlarning kesishgan nuqtasiga yo‘nalgan vektorni $\vec{c}$ orqali belgilaymiz.

Hosil bo‘lgan $\vec{c}$ vektor $\vec{a}$ va $\vec{b}$ vektorlarning yig‘indisi deyiladi va $\vec{a} + \vec{b}$ kabi yoziladi.

Bayon qilingan bu usul vektorlarni qo‘shishining **parallelogramm** usuli deyiladi.

Mustaqil ishlash uchun topshiriqlar.

(1) Vektorlarni qo‘shishining uchburchak usulini bayon qiling va bayon qilganlaringizni chizmada aks ettiring.

(2) Vektorlarni ayirishning uchburchak va parallelogram usullarini bayon qiling va bayon qilganlaringizni chizmada aks ettiring.

Koordinatalari bilan berilgan $\vec{a} = (a_1, a_2, a_3)$ va $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorlarning **yig‘indisi**

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

tenglik orqali topiladi.

$\vec{a} = (a_1, a_2, a_3)$ vektordan $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorning **ayirmasi**

$$\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

tenglik orqali topiladi.

Vektorni songa ko‘paytirish va vektorlarni qo‘shish amallari vektorlar ustida bajariladigan chiziqli amallar hisoblanadi. Bu chiziqli amallar quyidagi xossalarga ega:

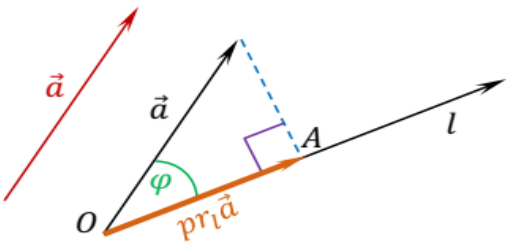
1^0. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$;

$2^0.\ (\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})=\vec{a}+\vec{b}+\vec{c};$
$3^0.\ \vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a};$
$4^0.\ \vec{a}+(-\vec{a})=\vec{0};$
$5^0.\ 1\cdot\vec{a}=\vec{a}\cdot 1=\vec{a};$
$6^0.\ (\alpha+\beta)\vec{a}=\alpha\vec{a}+\beta\vec{a};$
$7^0.\ \alpha(\vec{a}+\vec{b})=\alpha\vec{a}+\alpha\vec{b};$
$8^0.\ \alpha(\beta\vec{a})=(\alpha\beta)\vec{a}.$

Bayon qilingan xossalardagi barcha vektorlar yoki tekislik, yoki fazoning aynan bittasiga tegishli va ixtiyoriy, sonlar haqiqiy va ixtiyoriy deb olingan.

5.3. Vektorning o‘qdagi proeksiyasi

$\vec{a}$ vektor va l o‘q berilgan bo‘lsin. $\vec{a}$ vektorni shunday ko‘chiramizki, uning boshi l o‘qning boshi bilan ustma-ust tushsin. $\vec{a}$ vektorning oxiridan l o‘qqa perpendikulyar tushiramiz. Bu perpendikulyarning asosini A orqali belgilaymiz. Boshi O nuqtada, oxiri A nuqtada bo‘lgan vektorni $pr_l\vec{a}$ orqali belgilaymiz. $pr_l\vec{a}$ vektor $\vec{a}$ vektorning l o‘qdagi proeksiyasi deyiladi.



$\vec{a}$ vektor va l o‘q orasidagi burchak φ bo‘lsin. U holda $\vec{a}$ ning l dagi proeksiyasi

$$pr_l\vec{a}=\vec{a}\cos\varphi$$

tenglik orqali topiladi. Proeksiyaning o‘qdagi vaziyati (koordinatasi) $a_l=|\vec{a}|\cos\varphi$ tenglikdan topiladi.

$\vec{a}$ vektor va Ox, Oy, Oz o‘qlar orasidagi burchaklar mos ravishda α, β, γ bo‘lsin. U holda $\vec{a}$ vektorning koordinatalari ushbu

$$a_1=a_x=|\vec{a}|\cos\alpha,\qquad a_2=a_y=|\vec{a}|\cos\beta,\qquad a_3=a_z=|\vec{a}|\cos\gamma$$

tengliklar orqali topiladi.

Teskarisi, $\vec{a}$ vektor o‘zining koordinatalari bilan berilgan bo‘lsin: $\vec{a}=(a_1,\ a_2,\ a_3)$. U holda $\vec{a}$ vektor va Ox, Oy, Oz o‘qlar orasidagi mos α, β, γ burchaklar kosinuslari

$$\cos\alpha=\frac{a_1}{|\vec{a}|},\qquad \cos\beta=\frac{a_2}{|\vec{a}|},\qquad \cos\gamma=\frac{a_3}{|\vec{a}|}$$

tengliklar orqali topiladi. Bu $\cos\alpha$, $\cos\beta$, $\cos\gamma$ kosinuslar $\vec{a}$ vektorning yo‘naltiruvchi kosinuslari deyiladi. Yo‘naltiruvchi kosinuslar quyidagi ajoyib xossaga ega:

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

Mustaqil ishlash uchun topshiriq. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ tenglikni isbotlang.

Uzunligi 1 ga teng vektor birlik vektor deyiladi. Ravshanki, birlik vektorlar juda ko‘p.

Agar $\cos\alpha$, $\cos\beta$, $\cos\gamma$ lar biror $\vec{a}$ vektorning yo‘naltiruvchi kosinuslari bo‘lsa, $\vec{e}_a = (\cos\alpha, \cos\beta, \cos\gamma)$ vektor $\vec{a}$ vektor yo‘nalishidagi birlik vektor bo‘ladi.

Mos ravishda Ox, Oy, Oz o‘qlarining musbat yo‘nalishida yo‘nalgan $\vec{\imath} = (1,\ 0,\ 0)$, $\vec{\jmath} = (0,\ 1,\ 0)$, $\vec{k} = (0,\ 0,\ 1)$ birlik vektorlar $Oxyz$ Dekart koordinatalar sistemasining ortlari deyiladi. Har qanday $\vec{a} = (a_1,\ a_2,\ a_3)$ vektor $\{\vec{\imath},\ \vec{\jmath},\ \vec{k}\}$ ortlar orqali ushbu

$$\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$$

ko‘rinishda yagona usulda yoyiladi.

$\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ yozuv $\vec{a}$ vektorning $\{\vec{\imath},\ \vec{\jmath},\ \vec{k}\}$ ortlar bo‘yicha **yoyilmasi** deb ham yuritiladi.

Mustaqil ishlash uchun topshiriq.

1) $\vec{a} = (a_1,\ a_2,\ a_3)$ vektor haqiqatan ham $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ kabi yoyilmaga ega bo‘lishini va bu yoyilma yagona ekanligini isbotlang.

2) Vektorni songa ko‘paytirish, vektorlarni qo‘shish amallarining yoyilma shaklini yozing.

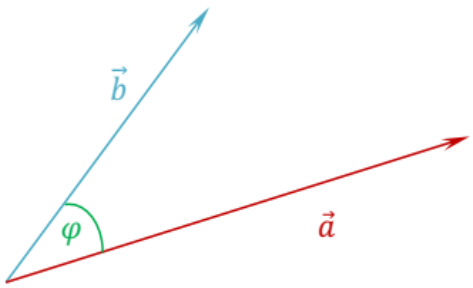
5.4. Ikkita vektorning skalyar ko‘paytmasi

$\vec{a}$ va $\vec{b}$ vektorlar va ular orasidagi φ burchak berilgan bo‘lsin. Ushbu

$$|\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$$

ko‘paytma $\vec{a}$ va $\vec{b}$ vektorlarning **skalyar ko‘paytmasi** deyiladi va $\vec{a}\vec{b}$ kabi belgilanadi. Demak,

$$\vec{a}\vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi.$$



Koordinatalari bilan berilgan $\vec{a} = (a_1, a_2, a_3)$ va $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorlarning skalyar ko‘paytmasi

$$\vec{a}\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

tenglik orqali topiladi.

Bu ikkita tenglikdan $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ va $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$ ekanligini e’tiborga olgan holda, berilgan $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak kosinusini hisoblashda qo‘llaniladigan ushbu

$$\cos\varphi = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

tenglik kelib chiqadi.

Skalyar ko‘paytma quyidagi xossalarga ega:

1^0. $\vec{a}\vec{b} = \vec{b}\vec{a}$;

2^0. $(\vec{a} + \vec{b})\vec{c} = \vec{a}\vec{c} + \vec{b}\vec{c}$;

3^0. $(k\vec{a})\vec{b} = k(\vec{a}\vec{b})$.

4^0. $\vec{a}\vec{a} \geq 0$. Bunda $\vec{a}\vec{a} = 0 \Leftrightarrow \vec{a} = \vec{0}$.

$\vec{a}$ vektorning $\vec{a}\vec{a}$ skalyar ko‘paytmasi $\vec{a}$ vektorning skalyar kvadrati deyiladi va $\vec{a}^2$ kabi belgilanadi. Koordinatalari bilan berilgan $\vec{a} = (a_1, a_2, a_3)$ vektorning skalyar kvadrati

$$\vec{a}^2 = a_1^2 + a_2^2 + a_3^2$$

tenglik orqali topiladi. $\vec{a}$ vektorning $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ uzunligi formulasini e’tiborga olsak, $|\vec{a}|^2 = \vec{a}^2$ yoki $|\vec{a}| = \sqrt{\vec{a}^2}$ tenglikni hosil qilamiz.

Mustaqil ishlash uchun topshiriq. Koshi-Bunyakovskiy tengsizligi deb ataluvchi ushbu

$$\left(\vec{a}\vec{b}\right)^2 \leq \vec{a}^2\vec{b}^2$$

tengsizlikni isbotlang.

Ko‘rsatma. Vektorlar orasidagi burchak kosinusi formulasidan va kosinusning qabul qiladigan qiymatlari to‘plamidan foydalaning.

$\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak 90^0 ga teng bo‘lsa, bunday vektorlar perpendikulyar (ortogonal) vektorlar deyiladi va $\vec{a} \perp \vec{b}$ kabi belgilanadi.

Mustaqil ishlash uchun topshiriqlar.

1) Noldan farqli ikkita $\vec{a}$ va $\vec{b}$ vektorlar uchun ushbu

$$\vec{a} \perp \vec{b} \Leftrightarrow \vec{a}\vec{b} = 0$$

perpendikulyarlik shartini isbotlang.

2) $Oxyz$ Dekart koordinatalar sistemasining $\vec{\imath} = (1,\ 0,\ 0)$, $\vec{\jmath} = (0,\ 1,\ 0)$, $\vec{k} = (0,\ 0,\ 1)$ ortlari uchun $\vec{\imath} \perp \vec{\jmath}$, $\vec{\imath} \perp \vec{k}$ va $\vec{\jmath} \perp \vec{k}$ ekanligini isbotlang.

3) $\vec{a} = (a_1, a_2, a_3)$ va $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorlarning $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ va $\vec{a} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ yoyilmalari hamda $\vec{\imath} \perp \vec{\jmath}$, $\vec{\imath} \perp \vec{k}$, $\vec{\jmath} \perp \vec{k}$ munosabatlardan foydalanib, bu vektorlar skalyar ko‘paytmasining

$$\vec{a}\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

formulasini isbotlang.

5.5. Ikkita vektorning vektor ko‘paytmasi

Avvalo, vektorni aniqlash degani – bu vektorning uzunligi va yo‘nalishini aniqlash ekanligini yana bir bor esga olamiz.

Bizning maqsadimiz ikkita vektor ustida qandaydir amallar bajarib, ko‘paytmada boshqa, **vektor ko‘paytma** deb ataluvchi uchinchi vektorni hosil qilishdan iborat. Uchinchi vektor (vektor ko‘paytma)ning uzunligini va yo‘nalishini berilgan ikkita vektorning uzunligi va yo‘nalishlaridan foydalanib topish talab etiladi.

Noldan farqli $\vec{a}$ va $\vec{b}$ vektorlar berilgan bo‘lsin. Bu degani, ularning $|\vec{a}|$ va $|\vec{b}|$ uzunliklari hamda yo‘nalishlari ma’lum degani. Yo‘nalishlar berilgan bo‘lsa, ular orasidagi $\varphi = \left(\widehat{\vec{a}, \vec{b}}\right)$ burchak ham aniqlangan bo‘ladi. $\vec{a}$ va $\vec{b}$ vektorlarning boshlarini ustma-ust tushadigan qilib joylashtiramiz.

$\vec{a}$ va $\vec{b}$ vektorlarning $\vec{a} \times \vec{b}$ **vektor ko‘paytmasi** deb, quyidagi shartlarni qanoatlantiruvchi $\vec{c}$ vektorga aytiladi:

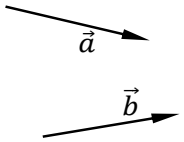
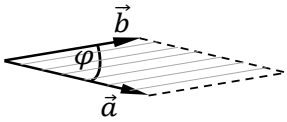
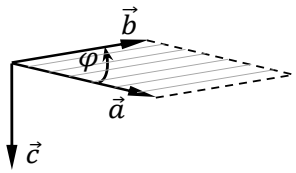
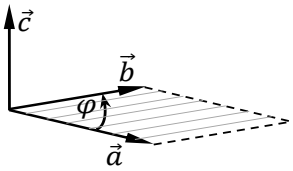
(V1) $|\vec{c}| = |\vec{a}||\vec{b}|\sin\varphi$ (*bu shart bilan $\vec{c}$ vektorning uzunligi aniqlandi*);

$\vec{c}$ vektorning boshini $\vec{a}$ va $\vec{b}$ vektorlarning umumiy boshiga shunday qo‘yamizki,

(V2) $\vec{c} \perp \vec{a}$ va $\vec{c} \perp \vec{b}$ hamda

(V3) $\vec{c}$ vektorning oxiridan kuzatganda $\vec{a}$ vektordan $\vec{b}$ vektorga eng qisqa burilish soat strelkasi yo‘nalishiga qarama-qarshi yo‘nalishda bo‘lsin (*2 va 3 shartlar orqali $\vec{c}$ vektorning yo‘nalishi aniqlandi*).

Keltirilgan ta’rifni ushbu jadvalda illyustratsiya qilamiz.

Berilgan $\vec{a}$ va $\vec{b}$ vektorlarning $\vec{a} \times \vec{b}$ vektor ko‘paytmasini quramiz.	Birinchi qadamda $\vec{a}$ va $\vec{b}$ vektorlarning boshlarini ustma-ust tushadigan qilib joylashtiramiz va $\vec{a} \times \vec{b}$ vektorning $\lvert\vec{a} \times \vec{b}\rvert$ uzunligini topamiz.	Ikkinchi qadamda fazoda uzunligi $\lvert\vec{a}\rvert\lvert\vec{b}\rvert\sin\varphi$ ga teng shunday $\vec{c}$ olamizki, $\vec{c}$ ning boshi $\vec{a}$ va $\vec{b}$ larning umumiy boshida bo‘lib, $\vec{c} \perp \vec{a}$ va $\vec{c} \perp \vec{b}$ bo‘lsin. Bunaqa vektorlar ikkita bo‘ladi. Uchinchi qadamda shu ikkita vektorlardan bittasi quyidagicha tanlanadi.	
 Vektor berilgan degani, uning uzunligi va yo‘nalishi ma’lum, degani ekanligini ta’kidlaymiz.	 Bu uzunlikning son qiymati $\vec{a}$ va $\vec{b}$ vektorlarga qurilgan parallelogrammning $S = \lvert\vec{a}\rvert\lvert\vec{b}\rvert\sin\varphi$ yuzasining son qiymatiga teng, ya’ni $\lvert\vec{a} \times \vec{b}\rvert = \lvert\vec{a}\rvert\lvert\vec{b}\rvert\sin\varphi$.	 $\vec{c}$ vektorning oxiridan kuzatganda $\vec{a}$ vektordan $\vec{b}$ vektorga eng qisqa burilish soat strelkasi yo‘nalishi bilan bir xil. Bu holda $\vec{a} \times \vec{b} \neq \vec{c}$.	 $\vec{c}$ vektorning oxiridan kuzatganda $\vec{a}$ vektordan $\vec{b}$ vektorga eng qisqa burilish soat strelkasi yo‘nalishiga qarama-qarshi. Bu holda $\vec{a} \times \vec{b} = \vec{c}$.

Eslatma. (V3) shartni qanoatlantiruvchi $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar **o‘ng uchlik tashkil etadi** deyiladi.

Agarda $\vec{a}$ va $\vec{b}$ vektorlarning birortasi nol-vektor bo‘lsa, $\vec{a} \times \vec{b} = 0$ bo‘ladi.

$\vec{a}$ va $\vec{b}$ vektorlarning $\vec{a} \times \vec{b}$ vektor ko‘paytmasini $[\vec{a}, \vec{b}]$ orqali ham belgilashadi.

Vektorlarning vektor ko‘paytmasi quyidagi xossalarga ega.

1^0. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

2^0. $\lambda(\vec{a} \times \vec{b}) = \big((\lambda\vec{a}) \times \vec{b}\big) = \big(\vec{a} \times (\lambda\vec{b})\big)$.

3^0. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Agarda $\vec{a}$ va $\vec{b}$ vektorlar kollinear bo‘lsa, u holda ravshanki, yo $\left(\widehat{\vec{a},\vec{b}}\right)=0$, yo $\left(\widehat{\vec{a},\vec{b}}\right)=\pi$ bo‘ladi. Bu holda $\sin\left(\widehat{\vec{a},\vec{b}}\right)=0$ bo‘lib, natijada vektorlarning **kollinearlik sharti** deb ataluvchi ushbu xossaga ega bo‘lamiz:

4^0. $\vec{a}\,||\,\vec{b} \Leftrightarrow \vec{a}\times\vec{b}=\vec{0}$.

Xususiy holda, har qanday $\vec{a}$ vektor uchun $\vec{a}\times\vec{a}=\vec{0}$ bo‘ladi.

Mustaqil ishlash uchun topshiriqlar. $Oxyz$ Dekart koordinatalar sistemasining $\vec{\imath}=(1,\ 0,\ 0)$, $\vec{\jmath}=(0,\ 1,\ 0)$, $\vec{k}=(0,\ 0,\ 1)$ ortlarinining vektor ko‘paytmalari uchun ushbu tengliklarni eslab qoling:

$$\vec{\imath}\times\vec{\jmath}=\vec{k},\qquad \vec{\jmath}\times\vec{k}=\vec{\imath},\qquad \vec{k}\times\vec{\imath}=\vec{\jmath},$$
$$\vec{\jmath}\times\vec{\imath}=-\vec{k},\qquad \vec{k}\times\vec{\jmath}=-\vec{\imath},\qquad \vec{\imath}\times\vec{k}=-\vec{\jmath},$$
$$\vec{\imath}\times\vec{\imath}=\vec{0},\qquad \vec{\jmath}\times\vec{\jmath}=\vec{0},\qquad \vec{k}\times\vec{k}=\vec{0},$$

Endi koordinatalari bilan berilgan $\vec{a}=\{a_x,a_y,a_z\}$ va $\vec{b}=\{b_x,b_y,b_z\}$ vektorlarning $\vec{a}\times\vec{b}$ vektor ko‘paytmasining koordinatalarini topish formulasini yozamiz:

$$\vec{a}\times\vec{b}=\left(\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix},-\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix},\begin{vmatrix}a_x & b_x\\ a_y & b_y\end{vmatrix}\right).$$

Diqqatingizni ikkita vektorning skalyar ko‘paytmasi qandaydir **son** ekaniga, ikkita vektorning vektor ko‘paytmasi esa qandaydir **vektor** ekaniga qarating.

5.6. Uchta vektorning aralash ko‘paytmasi

Ikkita vektorning skalyar ko‘paytmasi qandaydir son, ikkita vektorning vektor ko‘paytmasi esa qandaydir vektor bo‘ladi.

Noldan farqli $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlar berilgan bo‘lsin. $\vec{a}$ vektorning $\vec{b}\times\vec{c}$ vektor ko‘paytma bilan $\vec{a}(\vec{b}\times\vec{c})$ skalyar ko‘paytmasi **uchta vektorning aralash ko‘paytmasi** deyiladi.

Mustaqil ishlash uchun topshiriqlar.

1) $\vec{a}(\vec{b}\times\vec{c})$ ko‘paytma nima uchun aralash ko‘paytma deb atalishini izohlang.

2) aralash ko‘patma natijasida qanday kattalik hosil bo‘ladi?

A) vektor B) son C) yo‘nalish D) kattalik hosil bo‘lmaydi

3) $\vec{a}$ vektorning $\vec{b}\vec{c}$ skalyar ko‘paytma bilan $\vec{a}(\vec{b}\vec{c})$ ko‘paytmasini qanday atagan bo‘lar edingiz?

Agarda $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlardan birortasi nol-vektor bo‘lsa, u holda ularning aralash ko‘paytmasi nolga teng bo‘ladi: $\vec{a}(\vec{b}\times\vec{c})=0$.

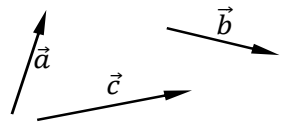
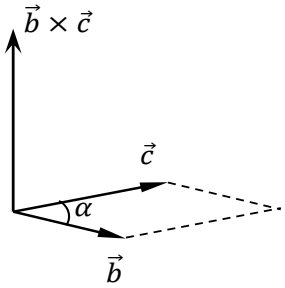
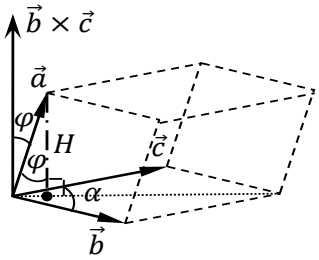
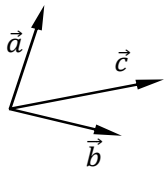
Parallel tekisliklarda yotuvchi vektorlar komplanar vektorlar deyiladi.

Mustaqil ishlash uchun topshiriq. Har qanday ikkita vektor komplanar bo‘lishini isbotlang.

Ko‘rsatma. Har qanday uchta nuqta uchun ular yotadigan tekislik har doim mavjudligidan foydalaning.

Komplanar bo‘lmagan uchta $\vec{a}$, $\vec{b}$, $\vec{c}$ vektorlar berilgan bo‘lsin. Ularning boshlarini bitta nuqtaga keltiramiz va bu vektorlarga asosi $\vec{b}$ va $\vec{c}$ vektorlarga qurilgan parallelogramm bo‘lgan parallelepiped quramiz. Bunda $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar o‘ng uchlik tashkil etsin.

Ushbu jadvalda aralash ko‘paytmaning illyustaritsiyasini keltiramiz.

 Berilgan komplanar bo‘lmagan vektorlar.	 $\vec{b}$ va $\vec{c}$ vektorlarga qurilgan parallelogramm yuzasi $$S = \lvert\vec{b}\rvert\lvert\vec{c}\rvert \sin\alpha.$$ ga teng. $$\lvert\vec{b}\times\vec{c}\rvert = S$$ ekanligini ta’kidlaymiz.	 Parallelepiped hajmi $V = SH$ ga teng, bu erda $H = \lvert\vec{a}\rvert\lvert\cos\varphi\rvert$ – balandlik. Shuning uchun $V = \lvert\vec{a}\rvert\lvert\vec{b}\times\vec{c}\rvert\lvert\cos\varphi\rvert$. Natijada, $$V = \lvert\vec{a}(\vec{b}\times\vec{c})\rvert.$$
 $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar o‘ng uchlik tashkil etsin.		

Shunday qilib, uchta vektorning aralash ko‘paytmasi modulining **geometrik talqini** shu vektorlarga qurilgan parallelepipedning hajmi ekan.

1) Agar $\vec{a}$, $\vec{b}$, $\vec{c}$ – o‘ng uchlik bo‘lsa,

$$V = \vec{a}(\vec{b}\times\vec{c})$$

bo‘ladi.

2) Agar $\vec{a}$, $\vec{b}$, $\vec{c}$ – chap uchlik bo‘lsa,

$$V = -\vec{a}(\vec{b}\times\vec{c})$$

bo‘ladi.

Uchta vektorning aralash ko‘paytmasi quyidagi xossalarga ega.

1^0. Siklik o‘rin almashtirishlarda aralash ko‘paytma o‘zgarmaydi, ya’ni

$$\vec{a}(\vec{b}\times\vec{c}) = \vec{b}(\vec{c}\times\vec{a}) = \vec{c}(\vec{a}\times\vec{b}).$$

2^0. $\vec{a}(\vec{b}\times\vec{c}) = -\vec{a}(\vec{c}\times\vec{b})$.

3^0. Koordinatali berilgan uchta $\vec{a} = (a_x, a_y, a_z)$, $\vec{b} = (b_x, b_y, b_z)$, $\vec{c} = (c_x, c_y, c_z)$ vektorlarning aralash ko‘paytmasi

$$\vec{a}(\vec{b}\times\vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

tenglik orqali topiladi.

4^0. Uchta vektorning **komplanarlik sharti**. $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar faqat va faqat $\vec{a}(\vec{b}\times\vec{c})=0$ bo‘lgandagina komplanar bo‘ladi.

Koordinatali berilgan uchta $\vec{a}=(a_x,a_y,a_z)$, $\vec{b}=(b_x,b_y,b_z)$, $\vec{c}=(c_x,c_y,c_z)$ vektorlar komplanar bo‘lishi uchun

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=0$$

bo‘lishi zarur va yetarli.

Mustahkamlash uchun mashqlar

Vektorlarga oid masalalar

1. Vektor uzunligini toping.

1) $\vec{a}=(3,\ 4)$ 2) $\vec{b}=(-2,\ 3)$ 3) $\vec{c}=(5,-3)$
4) $\vec{d}=(-\sqrt{3},-1)$ 5) $\vec{e}=(-3,-4,\ 12)$, 6) $\vec{f}=(8,-9,\ 12)$
7) $\vec{g}=(12,\ 21,-28)$, 8) $\vec{h}=(5,\ 12,\ 84)$ 9) $\vec{l}=(1,-1,\ \sqrt{2})$,
10) m va n lar ixtiyoriy sonlar bo‘lsin. $\vec{z}=(m^2-n^2,\ 2mn)$ vektor uzunligini toping.

2. Boshi va oxiri berilgan vektorning koordinatalari nimaga teng.

1) $A\left(-\frac{2}{5},\frac{3}{7}\right)$, $B\left(\frac{1}{3},\frac{4}{9}\right)$, $\overrightarrow{AB}=?$
2) $C(2\sqrt{3},-\sqrt{2})$, $D(\sqrt{3},\ \sqrt{2})$, $\overrightarrow{DC}=?$
3) $E(0{,}3;\ 3{,}1)$, $F(1{,}7;\ 4{,}8)$, $\overrightarrow{EF}=?$
4) $G(5,(37);\ 3,(7))$, $H(6,(43);\ 5,(70))$, $\overrightarrow{GH}=?$

3. Berilgan vektorning berilgan songa ko‘paytiring va hosil bo‘lgan vektorning uzunligini hisoblang.

1) $\vec{a}=(3,\ 4)$, $\lambda=5$ 2) $\vec{b}=(-2,\ 3)$, $\lambda=\frac{1}{\sqrt{13}}$
3) $\vec{c}=(5,-3)$, $\lambda=\frac{1}{\sqrt{34}}$ 4) $\vec{d}=(-\sqrt{3},-1)$, $\lambda=\frac{1}{2}$
5) $\vec{e}=(-3,-4,\ 12)$, $\lambda=-\frac{1}{13}$ 6) $\vec{f}=(8,-9,\ 12)$, $\lambda=\frac{1}{17}$
7) $\vec{g}=(12,\ 21,-28)$, $\lambda=\frac{1}{37}$ 8) $\vec{h}=(-5,\ 12,\ 84)$, $\lambda=\frac{1}{85}$
9) $\vec{l}=(1,-1,\ \sqrt{2})$, $\lambda=2$ 10) $\vec{m}=(33,\ 56,-72)$, $\lambda=\frac{1}{97}$.

4. Vektorlar yig‘indisi nimaga teng.

1) $\vec{a}=(3,4,5)$, $\vec{b}=(65,72,97)$ 2) $\vec{c}=(39,80,89)$, $\vec{d}=(5,12,13)$
3) $\vec{e}=(36,77,85)$, $\vec{f}=(8,15,17)$ 4) $\vec{g}=(13,84,85)$, $\vec{h}=(7,24,25)$
5) $\vec{l}=(48,55,73)$, $\vec{m}=(20,21,29)$ 6) $\vec{n}=(33,56,65)$, $\vec{p}=(12,35,37)$

7) $\vec{q} = (16, 63, 65)$, $\vec{r} = (9, 40, 41)$ 8) $\vec{s} = (28, 45, 53)$, $\vec{t} = (11, 60, 61)$.

5. Vektorning yo‘naltiruvchi kosinuslarini yozing.

1) $\vec{a} = (8, 15)$ 2) $\vec{b} = (-7, 24)$ 3) $\vec{c} = (20, -21)$
4) $\vec{d} = (-12, -35)$ 5) $\vec{e} = (9, 40, 41)$, 6) $\vec{f} = (28, 45, 53)$
7) $\vec{g} = (12,\ 21, -28)$, 8) $\vec{h} = (5,\ 12,\ 84)$ 9) $\vec{l} = \left(1, -1,\ \sqrt{2}\right)$,
10) m va n lar ixtiyoriy sonlar bo‘lsin. $\vec{z} = (m^2 - n^2,\ 2mn)$ vektorning yo‘naltiruvchi kosinuslarini toping.

6. Vektorlarning skalyar ko‘paytmasini hisoblang.

1) $\vec{a} = (2,\ -3)$, $\vec{b} = (-7,\ 5)$
2) $\vec{c} = \left(\frac{13}{14}, -\frac{1}{21}\right)$, $\vec{d} = \left(\frac{28}{13}, -105\right)$
3) $\vec{e} = \left(\sqrt{3},\ -\sqrt{2},\ 1\right)$, $\vec{f} = (\sin 60^0, \cos 45^0, -1)$
4) $\vec{g} = (12,\ 21, -84)$, $\vec{h} = \left(\frac{5}{6}, \frac{11}{21}, \frac{7}{21}\right)$
5) $\vec{l} = (\sin\alpha, \cos\alpha, -1)$, $\vec{m} = (\sin\alpha, \cos\alpha, 1)$

7. Vektorlar orasidagi burchak kosinusini toping.

1) $\vec{a} = (0, 4)$, $\vec{b} = (3,\ 0)$
2) $\vec{c} = (-33, 56)$, $\vec{d} = (56, 33)$
3) $\vec{e} = \left(6, 2\sqrt{3}\right)$, $\vec{f} = \left(6, 6\sqrt{3}\right)$
4) $\vec{g} = (5, 2)$, $\vec{h} = (3,\ 7)$.

8. Berilgan vektorlarga qurilgan parallelogramm yuzini hisoblang.

1) $\vec{a} = (-6{,}5;\ 7{,}2)$, $\vec{b} = (3{,}9;\ -8{,}0)$
2) $\vec{c} = (-3{,}6;\ -7{,}7)$, $\vec{d} = (1{,}3;\ 8{,}4)$
3) $\vec{e} = (1{,}3;\ 8{,}4;\ 8{,}5)$, $\vec{f} = (3{,}6;\ 7{,}7;\ 8{,}5)$
4) $\vec{g} = (1{,}6;\ -6{,}3;\ 6{,}5)$, $\vec{h} = (-3{,}3;\ 5{,}6;\ 6{,}5)$
5) $\vec{l} = (3, 4, -12)$, $\vec{l} = (-9, 12, 15)$.

9. Vektorlarning vektor ko‘paytmasini toping.

1) $\vec{a} = (-1,\ 0,\ 1)$, $\vec{b} = (3,\ 2,\ 0)$
2) $\vec{c} = (0,\ 3,\ 0)$, $\vec{d} = (1, -1,\ 1)$
3) $\vec{e} = (3,\ 4,\ 5)$, $\vec{f} = (6,\ 8,\ 10)$
4) $\vec{g} = (1,\ -6,\ 2)$, $\vec{h} = (-3,\ 3, -1)$
5) $\vec{l} = (3,\ 4,\ -1)$, $\vec{l} = (-9, -12,\ 0)$.

10. Uchta vektorning ko‘rsatilgan aralash ko‘paytmasini toping.

1) $\vec{a} = (-3,\ 3, -1)$, $\vec{b} = (1,\ -6,\ 2)$, $\vec{c} = (0,\ 3,\ 0)$, $\vec{a}\left(\vec{b} \times \vec{c}\right) = ?$ $\vec{c}\left(\vec{b} \times \vec{a}\right) = ?$

2), $\vec{d} = (1, -1,\ 1)$, $\vec{e} = (3,\ 4,\ 5)$, $\vec{f} = (6,\ 8,\ 10)$, $\vec{e}(\vec{f} \times \vec{d}) =?$ $\vec{d}(\vec{f} \times \vec{e}) =?$
5) $\vec{g} = (3,\ 2,\ 0)$, $\vec{h} = (-1,\ 0,\ 1)$, $\vec{l} = (3,\ 4,\ -1)$, $\vec{l}(\vec{h} \times \vec{g}) =?$ $\vec{g}(\vec{h} \times \vec{l}) =?$

11. 1) $\vec{a} = (1,\ 2,\ 3)$, $\vec{b} = (1, -1,\ 1)$, $\vec{c} = (2,\ 0, -1)$ vektorlarga qurilgan parallelepiped hajmini toping.

2) $ABCDD_1A_1B_1C_1$ parallelepipedning $A(1,\ 0, -2)$, $B(3,\ 1,\ 0)$, $D(0,\ 1, -3)$ va $A_1(3,\ 2,\ 0)$ uchlari berilgan. Uning hajmini hisoblang.

O`z-o`zini tekshirish uchun savollar

1. Vektor deb nimaga aytiladi?
2. Vektorlar ustida qanday chiziqli amallar bajarish mumkin?
3. Qanday vektorlar kollinear deyiladi?
4. Vektorning o‘qdagi proeksiyasi deb nimaga aytiladi?
5. Ikkita vektorning skalyar ko‘paytmasi deb nimaga aytiladi?
6. Ikkita vektorning vektor ko‘paytmasi deb nimaga aytiladi?
7. Ikkita vektorning vektor ko‘paytmasi qanday xossalarga ega?
8. Uchta vektorning aralash ko‘paytmasi deb nimaga aytiladi?
9. Uchta vektorning aralash ko‘paytmasi qanday xossalarga ega?

O`z-o`zini tekshirish uchun testlar

Vektorlarga oid testlar

1. Tekislikda berilagan $\vec{a} = (a_1,\ a_2,)$ vektorning uzunlikgini topish formulasini ko‘rsating.

a) $\vec{a} = \sqrt{a_1^2 + a_2^2}$ b) $\vec{a} = \sqrt{a_1^2 + a_2^2 + a_3^2}$
c) $\vec{a} = \sqrt{a_1^2 - a_2^2}$ b) $\vec{a} = \sqrt{a_1^2 - a_2^2 - a_3^2}$

2. Fazoda berilagan $\vec{a} = (a_1,\ a_2, a_3\ )$ vektorning uzunlikgini topish formulasini ko‘rsating.

a) $\vec{a} = \sqrt{a_1^2 + a_2^2}$ b) $\vec{a} = \sqrt{a_1^2 + a_2^2 + a_3^2}$
c) $\vec{a} = \sqrt{a_1^2 - a_2^2}$ d) $\vec{a} = \sqrt{a_1^2 - a_2^2 - a_3^2}$

3. Boshi $A(a_1,\ a_2,\ a_3)$ nuqtada, oxiri $B(b_1,\ b_2,\ b_3)$ nuqtada bo‘lgan $\vec{a}$ vektorning koordinatalarini topish formulasini ko‘rsating.

a) $\vec{a} = \overrightarrow{AB} = (b_1 - a_1,\ b_2 - a_2,\ b_3 - a_3)$
b) $\vec{a} = \overrightarrow{AB} = (b_1 + a_1,\ b_2 + a_2,\ b_3 - a_3)$
c) $\vec{a} = \overrightarrow{AB} = (b_1 - a_1,\ b_2 + a_2,\ b_3 - a_3)$
d) $\vec{a} = \overrightarrow{AB} = (b_1 + a_1,\ b_2 + a_2,\ b_3 + a_3)$

4. Boshi va oxiri ustma-ust tushadigan vektor ... deyiladi.

a) nol-vektor b) birlik vektor c) ort vektor d) bazis vektor

5. Uzunligi birga teng bo‘lgan vektor ... deyiladi.

a) nol-vektor b) birlik vektor c) ort vektor d) bazis vektor

6. $\vec{a} = (a_1, a_2, a_3)$ vektorning λ songa ko‘paytmasi … vektor bo‘ladi.

a) $\lambda\vec{a} = (-\lambda a_1, -\lambda a_2, -\lambda a_3)$ b) $\lambda\vec{a} = (\lambda a_1, \lambda a_2, \lambda a_3)$
c) $\lambda\vec{a} = (a_1, a_2, \lambda a_3)$ d) $\lambda\vec{a} = (\lambda a_1, a_2, a_3)$

7. Parallel to‘g‘ri chiziqlarda yotuvchi vektorlar ... vektorlar deyiladi.
a) kollinear b) nol-vektor c) birlik vektor d) ort vektor
8. $\vec{a} = (a_1, a_2, a_3)$ va $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorlarning kollinearlik sharti toping.
a) $\vec{a} \,||\, \vec{b} \Leftrightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}$ b) $\vec{a} \,||\, \vec{b} \Leftrightarrow \frac{b_1}{a_1} \neq \frac{b_2}{a_2} = \frac{b_3}{a_3}$
c) $\vec{a} \,||\, \vec{b} \Leftrightarrow \frac{b_1}{a_1} \neq \frac{b_2}{a_2} \neq \frac{b_3}{a_3})$ d) $\vec{a} \,||\, \vec{b} \Leftrightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} \neq \frac{b_3}{a_3})$
9. $\vec{a} = (a_1, a_2, a_3)$ va $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorlarning teng bo‘lish shartini toping.
a) $\vec{a} = \vec{b} \Leftrightarrow \begin{cases} a_1 = b_1, \\ a_2 = b_2, \\ a_3 = b_3. \end{cases}$ b) $\vec{a} = \vec{b} \Leftrightarrow \begin{cases} a_1 = -b_1, \\ a_2 = -b_2, \\ a_3 = -b_3. \end{cases}$
c) $\vec{a} = \vec{b} \Leftrightarrow \begin{cases} a_1 = b_1, \\ a_2 \neq b_2, \\ a_3 \neq b_3. \end{cases}$ d) $\vec{a} = \vec{b} \Leftrightarrow \begin{cases} a_1 = -b_1, \\ a_2 = b_2, \\ a_3 = b_3. \end{cases}$
10. Koordinatalari bilan berilgan $\vec{a} = (a_1, a_2, a_3)$ va $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorlarning yig‘indisini ko‘rsating.
a) $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ b) $\vec{a} + \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$
c) $\vec{a} + \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 + b_3)$ d) $\vec{a} + \vec{b} = (a_1 + b_1, a_2 - b_2, a_3 - b_3)$
11. Koordinatalari bilan berilgan $\vec{a} = (a_1, a_2, a_3)$ va $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorlarning ayirmasini ko‘rsating.
a) $\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$ b) $\vec{a} - \vec{b} = (a_1 - b_1, a_2 + b_2, a_3 + b_3)$
c) $\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 + b_3)$ d) $\vec{a} - \vec{b} = (a_1 - b_1, a_2 + b_2, a_3 - b_3)$
12. Agar $\vec{a}$ vektor va l o‘q orasidagi burchak φ bo‘lsa, u holda $\vec{a}$ ning l dagi proeksiyasini toping.
a) $pr_l\vec{a} = \vec{a}\cos\varphi$ b) $pr_l\vec{a} = \vec{a}\sin\varphi$
c) $pr_l\vec{a} = \vec{a}\,\mathrm{tg}\,\varphi$ d) $pr_l\vec{a} = \vec{a}\,\mathrm{ctg}\,\varphi$
14. Agar $\vec{a}$ vektor va Ox, Oy, Oz o‘qlar orasidagi burchaklar mos ravishda α, β, γ bo‘lsa, u holda $\vec{a}$ vektorning koordinatalari toping.
a) $a_1 = a_x = |\vec{a}|\cos\alpha$, $a_2 = a_y = |\vec{a}|\cos\beta$, $a_3 = a_z = |\vec{a}|\cos\gamma$
b) $a_1 = a_x = |\vec{a}|\sin\alpha$, $a_2 = a_y = |\vec{a}|\cos\beta$, $a_3 = a_z = |\vec{a}|\cos\gamma$
c) $a_1 = a_x = |\vec{a}|\sin\alpha$, $a_2 = a_y = |\vec{a}|\sin\beta$, $a_3 = a_z = |\vec{a}|\cos\gamma$
d) $a_1 = a_x = |\vec{a}|\sin\alpha$, $a_2 = a_y = |\vec{a}|\sin\beta$, $a_3 = a_z = |\vec{a}|\sin\gamma$
15. Agar $\vec{a} = (a_1,\ a_2,\ a_3)$ bo‘lsa, u holda $\vec{a}$ vektor va Ox, Oy, Oz o‘qlar orasidagi mos α, β, γ burchaklar kosinuslarini toping.
a) $\cos\alpha = \frac{a_1}{|\vec{a}|}$, $\cos\beta = \frac{a_2}{|\vec{a}|}$, $\cos\gamma = \frac{a_3}{|\vec{a}|}$
b) $\cos\alpha = \frac{a_1+a_2}{|\vec{a}|}$, $\cos\beta = \frac{a_2+a_1}{|\vec{a}|}$, $\cos\gamma = \frac{a_3+a_2}{|\vec{a}|}$
c) $\cos\alpha = \frac{2a_1}{|\vec{a}|}$, $\cos\beta = \frac{2a_2}{|\vec{a}|}$, $\cos\gamma = \frac{2a_3}{|\vec{a}|}$
d) $\cos\alpha = \frac{a_1-a_2}{|\vec{a}|}$, $\cos\beta = \frac{a_2-a_1}{|\vec{a}|}$, $\cos\gamma = \frac{a_3-a_2}{|\vec{a}|}$
16. $\vec{a}$ va $\vec{b}$ vektorlar va ular orasidagi φ burchak berilgan bo‘lsa, ularning skalyar ko‘paytmasi ko‘rsating.

a) $|\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$ b) $|\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi$

c) $\vec{a}\vec{b} = a_1b_1 + a_2b_2 - a_3b_3$ d) $\vec{a}\vec{b} = a_1b_1 - a_2b_2 - a_3b_3$

17. Koordinatalari bilan berilgan $\vec{a} = (a_1, a_2, a_3)$ va $\vec{b} = (b_1,\ b_2,\ b_3)$ vektorlarning skalyar ko‘paytmasi ko‘rsating.

a) $\vec{a}\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ b) $\vec{a}\vec{b} = a_1b_1 - a_2b_2 + a_3b_3$

c) $\vec{a}\vec{b} = a_1b_1 + a_2b_2 - a_3b_3$ d) $\vec{a}\vec{b} = a_1b_1 - a_2b_2 - a_3b_3$

18. $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak kosinusini topish formulasini ko‘rsating.

a) $\cos\varphi = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}$ b) $\cos\varphi = \frac{a_1b_1-a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}$

c) $\cos\varphi = \frac{a_1b_1-a_2b_2-a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}$ d) $\cos\varphi = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2-a_2^2+a_3^2}\cdot\sqrt{b_1^2-b_2^2+b_3^2}}$

19. $\vec{a} = \{a_x, a_y, a_z\}$ va $\vec{b} = \{b_x, b_y, b_z\}$ vektorlarning $\vec{a} \times \vec{b}$ vektor ko‘paytmasining koordinatalarini topish formulasini toping.

a) $\vec{a} \times \vec{b} = \left(\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}, -\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}, \begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}\right)$

b) $\vec{a} \times \vec{b} = \left(\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}, \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}, \begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}\right)$

c) $\vec{a} \times \vec{b} = \left(\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}, -\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}, -\begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}\right)$

d) $\vec{a} \times \vec{b} = \left(-\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}, -\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}, -\begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}\right)$

20. Koordinatali berilgan uchta $\vec{a} = (a_x, a_y, a_z)$, $\vec{b} = (b_x, b_y, b_z)$, $\vec{c} = (c_x, c_y, c_z)$ vektorlarning aralash ko‘paytmasi … tenglik orqali topiladi.

a) $\vec{a}(\vec{b} \times \vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$ b) $\vec{a}(\vec{b} \times \vec{c}) = \begin{vmatrix} -a_x & a_y & a_z \\ b_x & -b_y & b_z \\ c_x & c_y & -c_z \end{vmatrix}$

c) $\vec{a}(\vec{b} \times \vec{c}) = \begin{vmatrix} -a_x & a_y & a_z \\ -b_x & b_y & b_z \\ -c_x & c_y & c_z \end{vmatrix}$ d) $\vec{a}(\vec{b} \times \vec{c}) = \begin{vmatrix} a_x & a_y & -a_z \\ b_x & -b_y & b_z \\ -c_x & c_y & c_z \end{vmatrix}$

21. Koordinatali berilgan uchta $\vec{a} = (a_x, a_y, a_z)$, $\vec{b} = (b_x, b_y, b_z)$, $\vec{c} = (c_x, c_y, c_z)$ vektorlar komplanar bo‘lish shartini toping.

a) $\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 0$ b) $\begin{vmatrix} -a_x & a_y & a_z \\ b_x & -b_y & b_z \\ c_x & c_y & -c_z \end{vmatrix} = 0$

c) $\begin{vmatrix} -a_x & a_y & a_z \\ -b_x & b_y & b_z \\ -c_x & c_y & c_z \end{vmatrix} = 0$ d) $\begin{vmatrix} a_x & a_y & -a_z \\ b_x & -b_y & b_z \\ -c_x & c_y & c_z \end{vmatrix} = 0$

22. Agar $|\vec{a}| = 6$, $|\vec{b}| = 5$ va bu vektorlar orasidagi burchak 30^0 bo`lsa, u holda $\vec{a}$ va $\vec{b}$ vektorlarning vektor ko`paytmaning modulini toping.

a) 15 b) 5 c) 10 d) 12

23. Ikki vektorning skalyar ko`paytmasi nolga teng bo`lsa, uholda … bo`ladi.

a) ular o`zaro perependikulyar b) ular kollinear
c) ular o`zaro tub d) ulardan kamida bittasi nolga teng

24. $\vec{a} = -\vec{\imath} + \vec{\jmath}$ va $\vec{b} = \vec{\imath} + \vec{\jmath}$ vektorlar orasidagi burchakni toping.
a) 90^0 b)45^0 c) 30^0 d) 135^0

25. Agar $\vec{a} = (1, -4)$ va $\vec{b} = (-4,8)$ va bo`lsa, u holda $\vec{a} + \vec{b}$ vektorning modulini toping.
a) 5 b) 2 c) 1 d) 0

26. m ning qayday qiymatida $\vec{a} = (m, 1,2)$ va $\vec{b} = (2,4m, -3)$ vektorlaro`zaro perpendikulyar bo`ladi?
a) 1 b) 2 c) 7 d) 3

27. $\vec{a} = (4,3)$ vektorning modulini toping.
a) 5 b)2 c) 1 d) 3

28. Agar $A = (1,2,0)$ va $B = (3,2,1)$,bo`lsa, u holda $\overrightarrow{AB}$ vektorning uzunligini toping.
a) $\sqrt{5}$ b) 3 c) d) 4

29. Agar $|\vec{a}| = 3$, $|\vec{b}| = 4$ va bu vektorlar orasidagi burchak 60^0 bo`lsa, u holda $\vec{a}$ va $\vec{b}$ skalyar ko`paytmani toping.
a) 6 b) 3 c) 2 d) 0

30. Agar uchta vektor komplanar bo`lsa, u holda ularning aralash ko`paytmasi ... bo`ladi.
a) 0 b)-1 c) 1 d) 0,5

6-§. TEKISLIKDA ANALITIK GEOMETRIYA

6. Tekislikda analitik geometriya (nazariy qism)

Analitik geometriya asosan koordinatali $\mathbb{R}^3$ (yoki $\mathbb{R}^2$) fazoda, ya'ni ba'zi uch o`lchamli $Oxyz$ (yoki ikki o`lchamli Oxy) Dekart koordinatalari tizimida geometrik jismlarning tenglamalari bilan shug`ullanadi. Vaholanki, *geometrik ob'yekt*, jism (masalan, to`g`ri chiziq, tekislik, konus, giperbola, aylana va boshqalar) *ning tenglamasi* deganda ma'lum bir geometrik ob'yektga tegishli bo`lgan barcha nuqtalarning (x, y, z) koordinatalari o`rtasidagi munosabatni o`rnatadigan har qanday tenglama tushuniladi. Boshqacha aytganda, analitik geometriya geometrik jismlarni va ularning xususiyatlarini analitik, ya'ni ularning tenglamalarini tahlil qilish orqali o`rganadi.

Tekislikda Oxy koordinatalari tizimini qaraymiz. $F(x, y) = 0$ shakldagi tenglamani qanoatlantiradigan (x, y) tekislik nuqtalari to`plami *algebraik chiziq* (*egri chiziq*) deyiladi, bunda

$$F(x, y) = \sum_{i+j\le n} a_{ij}x^i y^j, \quad i, j = 0, 1, 2, \dots, n. \tag{1}$$

Agar $i + j = k$ bo`ladigan $a_{ij}$ koeffitsientlardan kamida bittasi nolga teng bo`lmasa va ko`rsatkichlari yig`indisi $i + j \ge k + 1$ bo`ladigan barcha koeffitsientlar nolga teng bo`lsa, u holda k son $F(x, y)$ ko`phadning *darajasi* hamda unga mos egri chiziqning *tartibi* deyiladi.

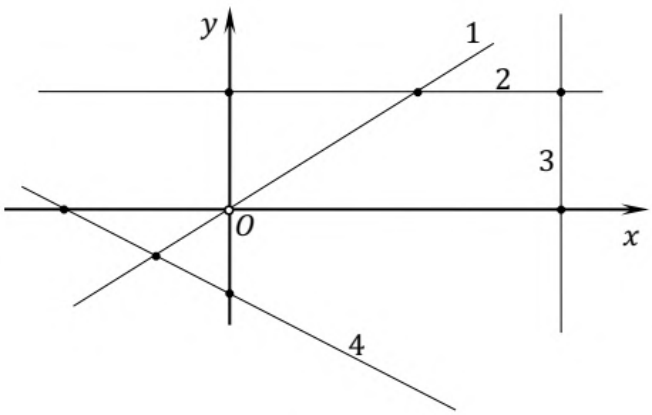
Agar egri chiziqning (x, y) nuqtalarining x va y koordinatalari parametr deb nomlangan bitta t o`zgaruvchiga bog`liq bo`lgan funksiyalar bilan berilgan ya'ni

$$\begin{cases} x = \varphi(t), \\ y = \psi(t), \end{cases} \tag{2}$$

bo`lsa, u holda bu egri chiziq *parametrik ravishda* berilgan deyiladi. Bunda (2) egri chiziqning *parametrik tenglamasi* deb yuritiladi.

6.1. Tekislikdagi to`g`ri chiziqning tenglamalari

Avvalo, to`g`ri chiziq matematikaning boshlang`ich tushunchasi ekanligini, ya'ni u avvaldan ma'lum deb olinishini va ta'rifsiz qabul qilinishini ta'kidlaymiz.



Rasmda Oxy tekisligidagi to`g`ri chiziqlar ko`rsatilgan. 1-chiziq koordinata boshi orqali o`tadi, 2-chiziq Ox o`qiga parallel, 3-chiziq $Oy$ o`qiga parallel bo`lib, 4-chiziq koordinata o`qlari bilan koordinata boshidan tashqarida kesishadi.

Bitta to`g`ri chiziqning barcha nuqtalari faqat va faqat shu nuqtalar ega bo`lgan xususiyatga ega bo`lishi kerak.

Shuning uchun har bir bunday xususiyat yagona to`g`ri chiziqning barcha nuqtalarini aniqlashi kerak. Analitik jihatdan bu xususiyat *to`g`ri chiziqning tenglamasi*dir.

Agar $i + j \leq 1$ bo`lsa, unda (1) tenglama tekislikda algebraik chiziqni beradi. Bunday holda, (1) ko`phad quyidagi shaklga ega:

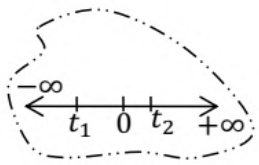
$$F(x, y) = a_{10}x + a_{01}y + a_{00}.$$

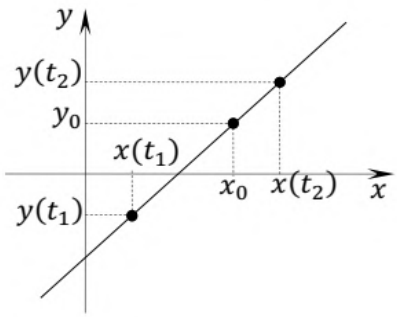
Ushbu

$$a_{10}x + a_{01}y + a_{00} = 0$$

tenglama to`g`ri chiziqning *umumiy tenglamasi* deb ataladi; bu tenglama odatda quyidagicha yoziladi:

$$ax + by + c = 0, \qquad \boxed{1}$$





To`g`ri chiziqning *parametrik tenglamasi*

$$\begin{cases} x = x_0 + \alpha t, \\ y = y_0 + \beta t, \end{cases} \qquad \boxed{2}$$

shaklga ega, bunda t – parametr, $-\infty < t + \infty$. $t = 0$ da

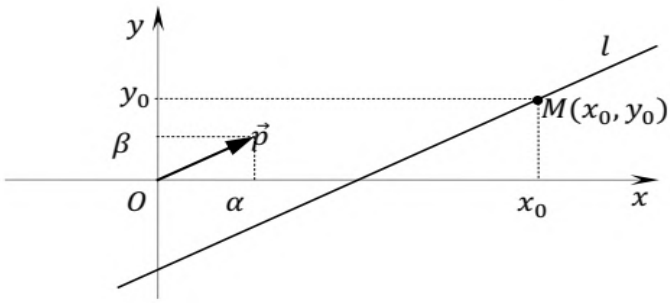
$x = x_0$ va

$y = y_0$ ekanligi ravshan.

Ushbu

$$\frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta}, \qquad \boxed{3}$$

tenglik (x_0, y_0) nuqtadan o`tgan to`g`ri chiziqning *kanonik tenglamasi* deyiladi.

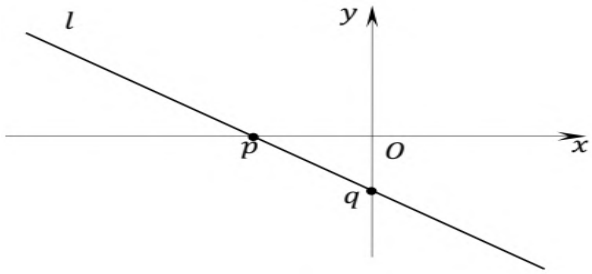


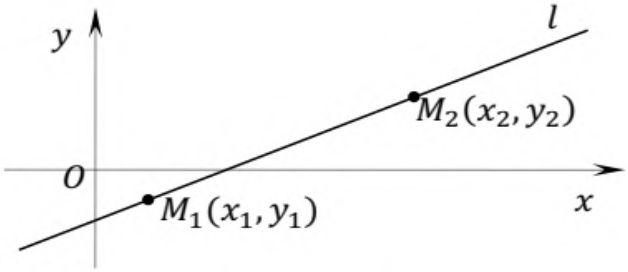
Koordinatalari α va β sonlardan iborat bo`lgan vektor, ya'ni $\vec{p} = \{\alpha, \beta\}$ vektor $\boxed{3}$ tenglama bilan berilgan $l$ chiziqning *yo`naltiruvchi vektori* deyiladi. $\boxed{3}$ tenglama bitta maxrajni nolga tenglashtirishga imkon beradi. Bunday holda, tegishli surat nolga tenglashtiriladi.

Ox va Oy o`qlarda koordinata boshidan uzunliklari mos ravishda $p$ va $q$ *kesmalar ajratuvchi* $l$ to`g`ri chiziq tenglamasi ushbu

$$\frac{x}{p} + \frac{y}{q} = 1 \qquad \boxed{4}$$

ko`rinishda bo`ladi.





Tekislikning turlicha $M_1(x_1, y_1)$ va $M_2(x_2, y_2)$ nuqtalari orqali o`tuvhci to`g`ri chiziq tenglamasi ushbu ko`rinishda bo`ladi

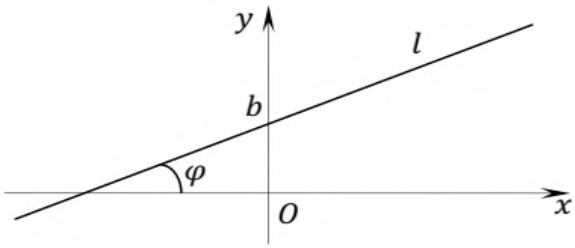
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}. \qquad \boxed{5}$$

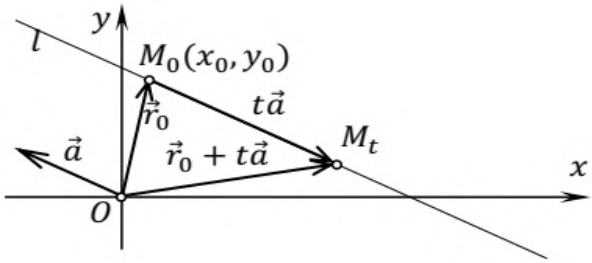
Ushbu

$$y = kx + b \qquad \boxed{6}$$

tenglama tekislikdagi *to`g`ri chiziqning burchak koeffitsiyenli tenglamasi* deyiladi.

$k = tg\varphi$ ekanligini ta'kidlaymiz. Bu yerda φ – to`g`ri chiziq bilan Ox o`qining musbat yo`nalishi orasidagi burchak.





$\vec{r}_0 = (x_0,\ y_0)$, $\vec{a} = (\alpha, \beta)$ lar berilgan vektorlar, $\vec{r} = \{x, y\}$ – o`zgaruvchi vektor, $t$ –parametr bo`lsin. Bunda $-\infty < t < +\infty$.

Ushbu

$$\vec{r} = \vec{r}_0 + \vec{a}t \qquad \boxed{7}$$

tenglama $(x_0,\ y_0)$ *nuqtadan o`tuvchi,* $\vec{a}$ *vektorga parallel to`g`ri chiziq tengamasi* deyiladi.

6.2. Tekislikdagi to`g`ri chiziqlarning o`zaro joylashuvi. Tekislikdagi to`g`ri chiziq va nuqta orasidagi masofa

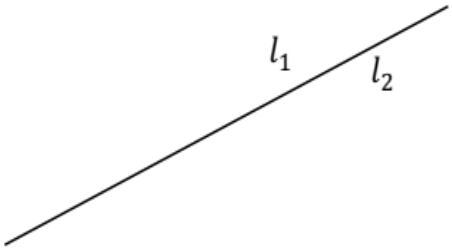
Tekislikda

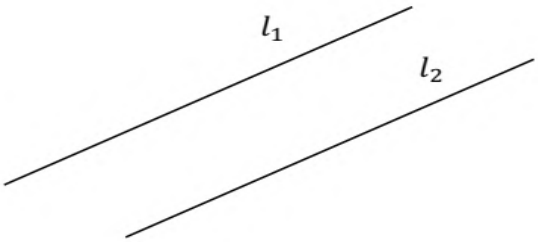
$$l_1: \qquad a_1x + b_1y + c_1 = 0,$$
$$l_2: \qquad a_2x + b_2y + c_2 = 0$$

to`g`ri chiziqlar berilgan bo`lsin. Quyidagi tasdiqlar o`rinli.

1. l_1 va l_2 to`g`ri chiziqlar ustma-ust tushishi uchun barcha mos koeffitsiyentlar va ozod hadlar nisbatlari teng bo`lishi zarur va yetarli, ya'ni

$$l_1 \equiv l_2 \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2};$$





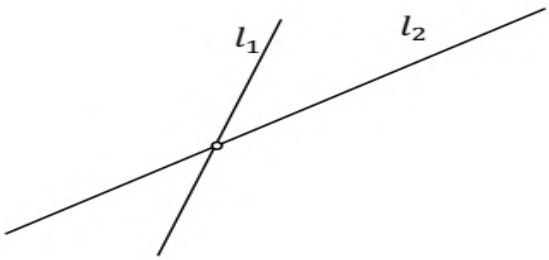
2. l_1 va l_2 to`g`ri chiziqlar (ustma-ust tushmaydigan) parallel bo`lishi uchun barcha mos koeffitsiyentlar nisbatlari teng bo`lishi zarur va yetarli, ya'ni

$$\big((l_1 \parallel l_2)\&(l_1 \not\equiv l_2)\big) \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2};$$

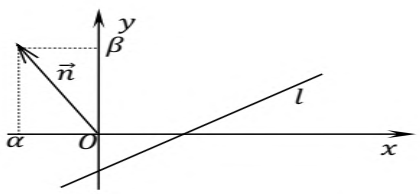
3. l_1 va l_2 to`g`ri chiziqlar bitta nuqtada kesishuvi uchun

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}.$$

bo`lishi zarur va yetarli.



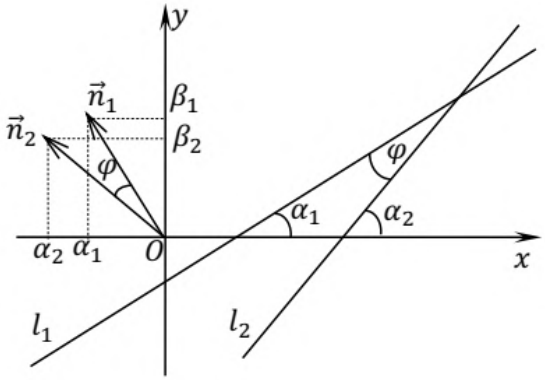
$ax + by + c = 0$ tenglama bilan berilgan l to`g`ri chiziqqa $\vec{n} = (a, b)$ vektor perpendikulyar bo`ladi va $l$ to`g`ri chiziqning normali deyiladi.



Oxy tekislikda

$$l_1: \quad a_1x + b_1y + c_1 = 0,$$
$$l_2: \quad a_2x + b_2y + c_2 = 0.$$

ikkita to`g`ri chiziq berilgan bo`lsin. U holda ularning normallari mos ravishda $\vec{n}_1 = (a_1, b_1)$ va $\vec{n}_2 = (a_2, b_2)$ vektorlar bo`ladi. Oydinki, l_1 va l_2 to`g`ri chiziqlar orasidagi burchak $\vec{n}_1 = (a_1, b_1)$ va $\vec{n}_2 = (a_2, b_2)$ vektorlar orasidagi burchakka teng, Shuning uchun



$$\cos(\widehat{l_1, l_2}) = \cos(\widehat{\vec{n}_1, \vec{n}_2}) = \frac{\vec{n}_1\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + a_2^2} \cdot \sqrt{b_1^2 + b_2^2}}.$$

Shunday qilib, l_1 va l_2 to`g`ri chiziqlar orasidagi $(\widehat{l_1, l_2})$ burchak

$$(\widehat{l_1, l_2}) = \arccos \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}}.$$

formula orqali hisoblanadi.

Agar to`g`ri chiziqlar

$$l_1: \quad y = k_1x + b,$$
$$l_2: \quad y = k_2x + b,$$

tenglamalar bilan berilgan bo`lsa, ular orasidagi $\varphi = (\widehat{l_1, l_2})$ burchak

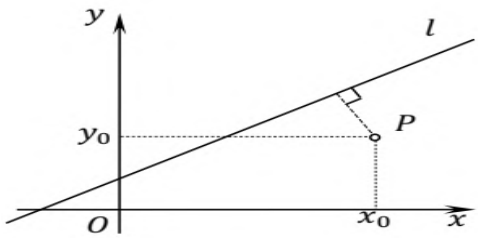
$$\varphi = arctg \frac{k_2 - k_1}{1 + k_1k_2}.$$

formula orqali topiladi.

Aytaylik,

$$l: \quad ax + by + c = 0$$

to`g`ri chiziq va $P(x_0, y_0)$ nuqta berilgan bo`lsin. $l$ to`g`ri chiziq va P nuqta orasidagi masofa

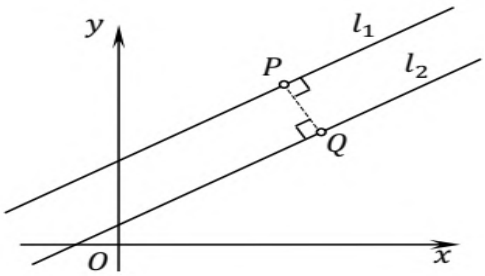


$$d(P, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

formula orqali aniqlanadi.

Oxy tekislikda berilgan ushbu to`g`ri chiziqlar parallel bo`lsin

$$l_1: \quad a_1x + b_1y + c_1 = 0,$$
$$l_2: \quad a_2x + b_2y + c_2 = 0.$$



Ular orasidagi $d(l_1, l_2)$ masofa quyidagicha topiladi:

$$d(l_1, l_2) = \frac{\left|a_2x - \frac{b_2}{b_1}(a_1x + c_1) + c_2\right|}{\sqrt{a_2^2 + b_2^2}}, \text{ yoki}$$

$$d(l_1, l_2) = \frac{\left|a_1x - \frac{b_1}{b_2}(a_2x + c_2) + c_1\right|}{\sqrt{a_1^2 + b_1^2}}.$$

6.3. Tekislikdagi ikkinchi tartibli egri chiziqlar: ellips, giperbola, parabola

Tekislikda Oxy Dekart koordinatalari sistemasini va undagi

$$\sum_{i+j\le n} a_{ij}x^i y^j = 0, \quad i, j = 0, 1, 2, \dots, n. \qquad (1)$$

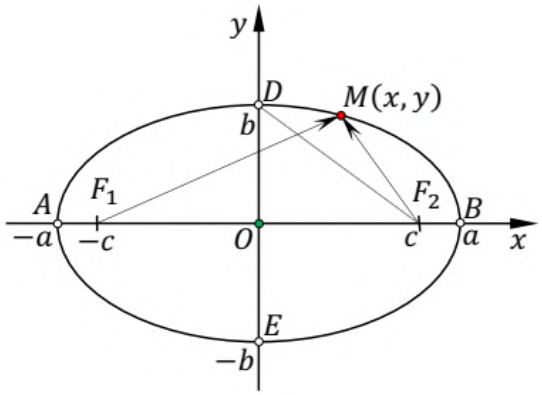
algebraik chiziqni qaraylik. $i + j \le 2$ hamda $i + j = 2$ bo`ladigan kamida bitta i, j juft topilsa, (1) tenglik tekislikdagi *ikkinchi tartibli algebraik egri chiziq*ni beradi. Bu holda (1) tenglama

$$a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0 \qquad (1')$$

ko`rinishda bo`ladi.

6.3.1. Ellips

Tekislikning berilgan ikkita F_1 va F_2 nuqtasigacha masofalari yig`indisi o`zgarmas va $2a$ ga teng bo`lgan nuqtalari to`plami *ellips* deyiladi $(a > 0)$. Bunda F_1 va F_2 nuqtalar ellipsning fokuslari deyiladi.



Ox o`q sifatida $F_1$ va $F_2$ nuqtalar orqali o`tuvchi, musbat yo`nalishi $F_1$ dan $F_2$ ga yo`nalish bilan aniqladigan o`q olinadi. Koordinata boshi deb $F_1F_2$ kesmaning o`rtasi olinadi. Oy o`qni $Ox$ o`qqa perpendikulyar qilib shunday yo`naltiranizki, $Ox$ dan $Oy$ ga eng qisqa burilish soat strelkasi harakati yo`nalishiga qarama-qarshi bo`lsin. U holda ellipsning *kanonik tenglamasi*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (2)$$

ko`rinishda bo`ladi.

Ellips koordinata o`qlarini *ellipsning uchlari* deb ataluvchi $A(-a, 0)$, $B(a, 0)$, $E(0, -b)$ va $D(0, b)$ nuqtalarda kesib o`tadi. Agar F_1 va F_2 fokuslar orasidagi masofani $2c$ orqali belgilasak, oydinki, $F_1(-c,\ 0)$, $F_2(c,\ 0)$ bo`ladi. $c < a$ bo`lishini ta'kidlaymiz. a, b va c sonlari orasida

$$b^2 = a^2 - c^2$$

munosabat o`rinli.

Kanonik tenglamasi (2) ko`rinishda bo`lgan ellips koordinata o`qlari va koordinata boshiga nisbatan simmetrik bo`ladi.

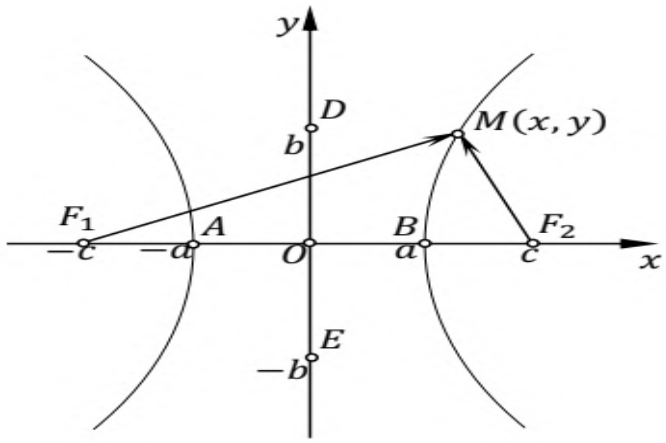
Uzunliklari $2a$ va $2b$ ga teng AB va ED kesmalar mos ravishda ellipsning *katta* va *kichik o`qlari*, $a$ va $b$ lar *katta* va *kichik yarim o`qlari* deyiladi.

Ellips shaklini uning ekstsentriti deb ataluvchi $e = \frac{c}{a}$ kattalik orqali aniqlash mumkin. Har doim $0 < e < 1$ bo`ladi. Ekstsenrisitet 1 ga qancha yaqin bo`lsa, ellips Ox o`qi bo`yicha shuncha cho`zilgan shaklga ega bo`ladi. Ekstsentrisitet nolga qancha yaqin bo`lsa, u aylanaga shuncha yaqin bo`ladi.

6.3.2. Giperbola

Tekislikning berilgan ikkita F_1 va F_2 nuqtasigacha masofalari ayirmasining moduli o`zgarmas va $2a$ ga teng bo`lgan nuqtalari to`plami *giperbola* deyiladi $(a > 0)$. Bunda F_1 va F_2 nuqtalar giperbolaning fokuslari deyiladi.

Ox o`q sifatida $F_1$ va $F_2$ nuqtalar orqali o`tuvchi, musbat yo`nalishi $F_1$ dan $F_2$ ga yo`nalish bilan aniqladigan o`q olinadi. Koordinata boshi deb $F_1F_2$ kesmaning o`rtasi olinadi. Oy o`qni $Ox$ o`qqa perpendikulyar qilib shunday yo`naltiranizki, $Ox$ dan $Oy$ ga eng qisqa burilish soat strelkasi harakati yo`nalishiga qarama-qarshi bo`lsin. U holda giperbolaning *kanonik tenglamasi*



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (3)$$

ko`rinishda bo`ladi.

Agar F_1 va F_2 fokuslar orasidagi masofani $2c$ orqali belgilasak, oydinki, $F_1(-c,\ 0)$, $F_2(c,\ 0)$ bo`ladi. $c > a$ bo`lishini ta'kidlaymiz. a, b va c sonlari orasida $c^2 = a^2 + b^2$ munosabat o`rinli.

Kanonik tenglamasi (3) ko`rinishda bo`lgan giperbola koordinata o`qlari va koordinata boshiga nisbatan simmetrik bo`ladi. U Ox o`qi bilan ikkita $A(-a, 0)$ va $B(a, 0)$ nuqtalarda kesishadigan ikkita qism (shox)dan iborat. Bu nuqtalar ellipsning uchlari deyiladi. Uzunligi $2a$ ga teng $AB$ kesma giperbolaning *haqiqiy o`qi* deyiladi. $E(0, -b)$ va $D(0, b)$ nuqtalar giperbolaning *mavhum uchlari* deyiladi. Uzunligi $2b$ ga teng DE kesma giperbolaning *mavhum o`qi* deyiladi.

Giperbola shaklini uning ekstsentriti deb ataluvchi $e = \frac{c}{a}$ kattalik tavsiflaydi. Har doim $e > 1$ bo`ladi. Ekstsenrisitet 1 ga qancha yaqin bo`lsa, giperbolaning shoxlari Ox o`qiga shuncha siqilgan bo`ladi. Ekstsentrisitet cheksizga yaqinlashsa, giperbola shoxlari Oy o`qiga parallel ikkita to`g`ri chiziqqa shuncha yaqinlashadi.

Giperbola tenglamalari

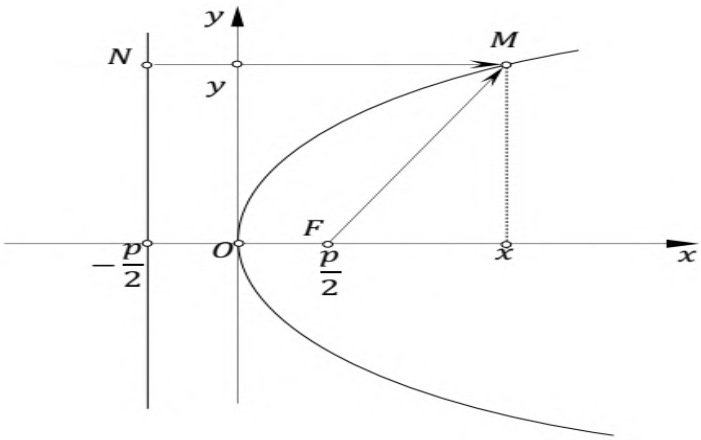
$$y = \pm\frac{b}{a}x.$$

ko`rinishda bo`lgan asimptotalarga ega. Koordinata boshidan uzoqlashganda giperbola shoxlari shu asimptotalarga yaqinlashadi.

6.3.3. Parabola

Tekislikning berilgan to`g`ri chiziqdan va berilgan nuqtadan bir xil uzoqlikda joylashgan nuqtalari to`plami *parabola* deyiladi. Berilgan to`g`ri chiziq parabolaning *direktrisasi*, berilgan nuqta parabolaning *fokusi* deyiladi.

Ox o`q sifatida fokus orqali o`tuvchi, direktrisa bilan perpendikulyar kesishuvchi, musbat yo`nalishi direktrisadan fokusga tomon yo`nalish bilan ustma-ust tushadigan o`q olinadi (bu holda $Ox$ o`q parabolaning *optik o`qi* deyiladi). Direktrisadan fokusgacha masofani $p$ orqali belgilaymiz va uni parabolaning *parametri* deymiz. Koordinata boshi sifatida $Ox$ o`qida direktrisa va fokus bilan teng uzoqlikdagi O nuqta olinadi. Oy o`qni $Ox$ o`qqa perpendikulyar qilib shunday yo`naltiranizki, $Ox$ dan $Oy$ ga eng qisqa burilish soat strelkasi harakati yo`nalishiga qarama-qarshi bo`lsin. U holda parabolaning *kanonik tenglamasi*



$$y^2 = 2px. \qquad (4)$$

ko`rinishda bo`ladi. Direktrisa tenglamasi $x = -\frac{p}{2}$ ko`rinishda, fokus koordinatalari $F\left(\frac{p}{2},\ 0\right)$ bo`ladi.

Tenglamasi (4) ko`rinishdagi parabola $Ox$ o`qiga simmetrik va koordinata boshidan o`tadi.

6.4. Текисликдаги иккинчи тартибли эгри чизиқларни каноник ҳолга келтириш

Ushbu

$$a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0. \qquad (1')$$

tenglama tekislikdagi ikkinchi tartibli egri chiziqning *umumiy* tenglamasi deyiladi.

Egri chiziq tipini egri chiziqning *determinanti* deb ataluvchi ushbu

$$\Delta = \begin{vmatrix} a_{20} & \frac{a_{11}}{2} \\ \frac{a_{11}}{2} & a_{02} \end{vmatrix}$$

determinant orqali darhol aniqlash mumkin.

Egri chiziq: agar $\Delta > 0$ bo`lsa, *elliptik tipdagi egri chiziq*; agar $\Delta < 0$ bo`lsa, *giperbolik tipdagi egri chiziq*; agar $\Delta = 0$ bo`lsa, *parabolik tipdagi egri chiziq* bo`ladi.

Boshqa, maxsus holatlar ham bo`lishi mumkin. Umuman olganda, ikkinchi tartibli egri chiziqlarni ushbu ko`rinishlardan biriga keltirish mumkin:

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ellips);

2. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$ (mavhum ellips);
3. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (giperbola);
4. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ (kesishuvchi to`g`ri chiziqlar jufti);
5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ (kesishuvchi mavhum to`g`ri chiziqlar jufti);
6. $y^2=2px$ (parabola);
7. $y^2-a^2=0$ ($a>0$, parallel to`g`ri chiziqlar jufti);
8. $y^2+a^2=0$ ($a>0$, parallel mavhum to`g`ri chiziqlar jufti);
9. $y^2=0$ (ustma-ust tushadigan to`g`ri chiziqlar jufti).

6.4.1.-misol. Koordinatalar o`qini parallel ko`chirish va burishlar orqali

$$3x^2-2xy+3y^2+4x+4y-4=0$$

egri chiziqni kanonik ko`rinishga keltiring.

Yechish. Avvalo egri chiziqning tipini aniqlaymiz. Ushbu determinantni hisoblaymiz

$$\Delta=\begin{vmatrix}3 & -1\\ -1 & 3\end{vmatrix}=9-1=8.$$

$\Delta>0$ bo`lgani uchun berilgan tenglama elliptik tipdagi chiziq tenglamasi bo`ladi.

Koordinata o`qlarini

$$x=x'+a,\ y=y'+b$$

kabi parallel ko`chirib,

$$3(x'+a)^2-2(x'+a)(y'+b)+3(y'+b)^2+4(x'+a)+4(y'+b)-4=0$$

ga ega bo`lamiz. Almashrtirishlar bajargach,

$$3x'^2-2x'y'+3y'^2+(6a-2b+4)x'+(-2a+6b+4)y'+$$
$$+3a^2-2ab+3b^2+4a+4b-4=0$$

bo`ladi. Yangi koordinatalar sistemasini shunday tanlaymizki,

$$\begin{cases}6a-2b+4=0,\\ -2a+6b+4=0\end{cases}$$

bo`lsin. Bu sistemani yechib, $a=-1$, $b=-1$ ni topamiz. Demak, yangi $O'x'y'$ koordinatalar sistemasining boshi $O'(-1,-1)$ nuqtada bo`ladi. Bu yangi sistemada egri chiziq tenglamasi

$$3x'^2-2x'y'+3y'^2-8=0$$

ko`rinishda bo`ladi.

Endi koordinata o`qlarini

$$\begin{cases}x'=x''\cos\varphi+y''\sin\varphi\,,\\ y'=y''\cos\varphi-x''\sin\varphi\,.\end{cases}$$

formulalar orqali buramiz. Bu ifodalarni $3x'^2-2x'y'+3y'^2-8=0$ tenglamaga qo`yib,

$$3(x''\cos\varphi+y''\sin\varphi)^2-2(x''\cos\varphi+y''\sin\varphi)(y''\cos\varphi-x''\sin\varphi)+$$
$$+3(y''\cos\varphi-x''\sin\varphi)^2-8=0$$

tenglikka ega bo`lamiz. Koeffitsiyentlarni almashtirib, ushbuni hosil qilamiz:

$$(3\cos^2\varphi+2\cos\varphi\sin\varphi+3\sin^2\varphi)x''^2+$$
$$(3\sin^2\varphi-2\cos\varphi\sin\varphi+3\cos^2\varphi)y''^2-$$

$$-2(\cos^2\varphi - \sin^2\varphi)x''y'' - 8 = 0.$$

Burishning φ burchagini shunday tanlaymizki, $x''y''$ ko`paytma oldidagi koeffitsiyent nolga teng, ya'ni $cos^2\varphi - sin^2\varphi = 0$ bo`lsin. Bu tenglikni qanoatlantiruvchi eng kichik φ burchak $\frac{\pi}{4}$ ga teng, ya'ni $\varphi = \frac{\pi}{4}$. Natijada egri chiziq tenglamasi

$$4x''^2 + 2y''^2 - 8 = 0,$$

$$\frac{x''^2}{(\sqrt{2})^2} + \frac{y''^2}{2^2} = 1.$$

ko`rinishga keladi. Bu – ellipsning $O'x''y''$ koordinatalar sistemasidagi kanonik tenglamasidir.

Mustahkamlash uchun mashqlar

Tekislikda analitik geometriyaga oid masalalar

1. Uchlari A, B, C nuqtalarda bo'lgan uchburchakning a) AC tomonining uzunligi; b) AC tomonining tenglamasini; c) B uchidan tushirilgan balandlik uzunligi va tenglamasi; d) AB va AC tomonlar orasidagi burchakni toping.

1. $A(4,5)$, $B(8,0)$, $C(-4,-5)$
2. $A(2,3)$, $B(2,6)$, $C(1,5)$
3. $A(8,0)$, $B(-3,0)$, $C(4,1)$
4. $A(6,2)$, $B(2,0)$, $C(0,0)$
5. $A(2,2)$, $B(6,5)$, $C(0,5)$
6. $A(3,-5)$, $B(3,2)$, $C(4,-5)$
7. $A(8,2)$, $B(-3,3)$, $C(4,-1)$
8. $A(-6,2)$, $B(-2,0)$, $C(0,0)$
9. $A(-2,8)$, $B(6,5)$, $C(0,5)$
10. $A(5,-5)$, $B(6,2)$, $C(4,-5)$

2. To'g'ri chiziqlar orasidan parallel va perpendikulyar to'g'ri chiziqlarni aniqlang.

1. $3x - 2y + 7 = 0,\ 2x + 3y - 6 = 0,\ 6x + 4y - 9 = 0,\ 6x + 4y - 5 = 0$
2. $5x - 2y + 7 = 0,\ 2x + 5y - 6 = 0,\ 2x + 5y - 9 = 0,\ 2x + 5y - 5 = 0$
3. $4x - 3y + 7 = 0,\ 3x + 4y - 6 = 0,\ 2x + 4y + 9 = 0,\ 2x + 4y + 6 = 0$
4. $3x - 2y + 7 = 0,\ 3x - 2y + 6 = 0,\ 6x + 4y - 9 = 0,\ 4x - 6y + 3 = 0$
5. $3x - 2y + 7 = 0,\ 2x + 3y - 6 = 0,\ 6x + 4y - 9 = 0,\ 6x + 4y - 5 = 0$

3. Quyidagi ikkinchi tartibli chiziqlarning turini aniqlab, fokusi, direktrisasi va asimptotalarini toping, uning grafigini yasang.

1. $16x^2 - 2y + 7 = 0$
2. $16x^2 - 9y^2 - 144 = 0$
3. $25x^2 - 4y^2 - 16 = 0$
4. $x^2 - 6x + 9y^2 - 144 = 0$
5. $25x^2 - 20x + 4y^2 - 16 = 0$
6. $2x - y^2 + 4 = 0$
7. $25x^2 + 9y^2 - 255 = 0$
8. $\frac{x^2}{25} + \frac{y^2}{9} - 1 = 0$

O`z-o`zini tekshirish uchun savollar

1. Tekislikdagi to`g`ri chiziqning qanday tenglamalarini bilasiz?
2. Tekislikdagi to`g`ri chiziqlarning o`zaro joylashuvi.
3. Tekislikdagi to`g`ri chiziq va nuqta orasidagi masofani topish formulasini yozing.
4. Tekislikdagi qanday ikkinchi tartibli egri chiziqlarni bilasiz
5. Ellips, giperbola va parabolalarning kanonik tenglamasini yozing.
6. Tekislikdagi ikkinchi tartibli egri chiziqlarni qanday kanonik holga keltiriladi?

O`z-o`zini tekshirish uchun testlar

Tekislikda analitik geometriyaga oid testlar

1. Tekislikda Oxy koordinatalari tiziminida $F(x,y)=0$ shakldagi tenglamani qanoatlantiradigan (x,y) tekislik nuqtalari to`plami nima deyiladi?

a) algebraik chiziq (egri chiziq) deyiladi. b) aylana chiziq deyiladi.
c) konus deyiladi. d) shar deyiladi.

2. $F(x,y)=\sum_{i+j\le n} a_{ij}x^i y^j$, $i,j=0,1,2,\dots,n$ ko`phadda agar $i+j=k$ bo`ladigan a_{ij} koeffitsientlardan kamida bittasi nolga teng bo`lmasa va ko`rsatkichlari yig`indisi $i+j\ge k+1$ bo`ladigan barcha koeffitsientlar nolga teng bo`lsa, u holda $k$ son $F(x,y)$ ko`phadning ...

a) darajasi hamda unga mos egri chiziqning tartibi deyiladi.
b) koeffitsienti hamda unga mos egri chiziqning koeffitsienti deyiladi.
c) uzunligi hamda unga mos egri chiziqning tartibi uzunligi deyiladi.
d) koordinatalari hamda unga mos egri chiziqning koordinatalari deyiladi.

3. Agar egri chiziqning (x,y) nuqtalarining x va y koordinatalari parametr deb nomlangan bitta t o`zgaruvchiga bog`liq bo`lgan funksiyalar $\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases}$ bilan berilgan bo`lsa, u holda bu egri chiziq

a) parametrik ravishda berilgan deyiladi va egri chiziqning parametrik tenglamasi deb ataladi.
b) umumiy ko`rinishda berilgan deyiladi va egri chiziqning umumiy tenglamasi deb ataladi.
c) koordinatalar o`qlaridan ajratgan kesma ko`rinishda berilgan deyiladi va egri chiziqning koordinatalar o`qlaridan ajratgan kesma tenglamasi deb ataladi.
d) burchak koeffitsiyenli ko`rinishda berilgan deyiladi va egri chiziqning burchak koeffitsiyenli tenglamasi deb ataladi.

4. $F(x,y)=\sum_{i+j\le n} a_{ij}x^i y^j$, $i,j=0,1,2,\dots,n$ tenglamada $i+j\le 1$ bo`lsa, u holda bu tenglama qanday ko`rinishga keladi?

a) $F(x,y)=a_{10}x+a_{01}y+a_{00}$. b) $F(x,y)=a_{10}x^2+a_{01}y+a_{00}$.
c) $F(x,y)=a_{10}x^2+a_{01}y^2+a_{00}$. d) $F(x,y)=a_{10}x+a_{01}y^2+a_{00}$.

5. Ushbu $a_{10}x + a_{01}y + a_{00} = 0$ tenglama to`g`ri chiziqning qanday tenglamasi deb ataladi va bu tenglama odatda qanday yoziladi?

a) umumiy tenglamasi deb ataladi va odatda $ax + by + c = 0$ ko`rinishda yoziladi.

b) burchak koeffitsiyenli tenglamasi deb ataladi va odatda $ax + by + c = 0$ ko`rinishda yoziladi.

c) parametrik tenglamasi deb ataladi va odatda $ax + by + c = 0$ ko`rinishda yoziladi.

d) parametrik tenglamasi deb ataladi va odatda $ax + b + c = 0$ ko`rinishda yoziladi.

6. To`g`ri chiziqning parametrik tenglamasini ko`rsating.

a) $\begin{cases} x = x_0 + \alpha t, \\ y = y_0 + \beta t, \end{cases} -\infty < t + \infty$ b) $ax + b + c = 0$

c) $\frac{x-x_0}{\alpha} = \frac{y-y_0}{\beta}$ d) $y = kx + b$

7. $\vec{p} = \{\alpha, \beta\}$ vektorga parallel va (x_0, y_0) nuqtadan o`tgan to`g`ri chiziqning kanonik tenglamasini ko`rsating.

a) $\begin{cases} x = x_0 + \alpha t, \\ y = y_0 + \beta t, \end{cases} -\infty < t + \infty$ b) $ax + b + c = 0$

c) $\frac{x-x_0}{\alpha} = \frac{y-y_0}{\beta}$ d) $y = kx + b$

8. Ox va Oy o`qlarda koordinata boshidan uzunliklari mos ravishda $p$ va $q$ kesmalar ajratuvchi $l$ to`g`ri chiziq tenglamasini ko`rsating.

a) $\frac{x}{p} + \frac{y}{q} = 1$ b) $ax + b + c = 0$

c) $\frac{x-x_0}{\alpha} = \frac{y-y_0}{\beta}$ d) $y = kx + b$

9. To`g`ri chiziqning burchak koeffitsiyenli tenglamasini ko`rsating.

a) $\begin{cases} x = x_0 + \alpha t, \\ y = y_0 + \beta t, \end{cases} -\infty < t + \infty$ b) $ax + b + c = 0$

c) $\frac{x-x_0}{\alpha} = \frac{y-y_0}{\beta}$ d) $y = kx + b$

10. To`g`ri chiziqning umumiy tenglamasini ko`rsating.

a) $\begin{cases} x = x_0 + \alpha t, \\ y = y_0 + \beta t, \end{cases} -\infty < t + \infty$ b) $ax + b + c = 0$

c) $\frac{x-x_0}{\alpha} = \frac{y-y_0}{\beta}$ d) $y = kx + b$

11. Tekislikning turlicha $M_1(x_1, y_1)$ va $M_2(x_2, y_2)$ nuqtalari orqali o`tuvhci to`g`ri chiziq tenglamasini ko`rsating.

a) $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$ b) $ax + b + c = 0$

c) $\frac{x-x_0}{\alpha} = \frac{y-y_0}{\beta}$ d) $y = kx + b$

12. Tekislikda l_1: $a_1x + b_1y + c_1 = 0$ va l_2: $a_2x + b_2y + c_2 = 0$ to`g`ri chiziqlar ustma-ust tushish sharitni ko`rsating.

a) $l_1 \equiv l_2 \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ b) $l_1 \equiv l_2 \Leftrightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2} = \frac{c_1}{c_2}$

c) $l_1 \equiv l_2 \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ d) $l_1 \equiv l_2 \Leftrightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

13. Tekislikda l_1: $a_1x + b_1y + c_1 = 0$ va l_2: $a_2x + b_2y + c_2 = 0$ to`g`ri chiziqlar parallel bo`lish sharitni ko`rsating.

a) $\left((l_1 \parallel l_2)\&(l_1 \not\equiv l_2)\right) \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

b) $\left((l_1 \parallel l_2)\&(l_1 \not\equiv l_2)\right) \Leftrightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2} = \frac{c_1}{c_2}$

c) $\left((l_1 \parallel l_2)\&(l_1 \not\equiv l_2)\right) \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

d) $\left((l_1 \parallel l_2)\&(l_1 \not\equiv l_2)\right) \Leftrightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

14. Tekislikda l_1: $a_1x + b_1y + c_1 = 0$ va l_2: $a_2x + b_2y + c_2 = 0$ to`g`ri chiziqlar bitta nuqtada kesish sharitni ko`rsating.

a) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ b) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

c) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ d) $\frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

15. $ax + by + c = 0$ tenglama bilan berilgan l to`g`ri chiziqqa perpendikulyar vektorni ko`rsating.

a) $\vec{n} = (a, b)$ b) $\vec{n} = (b, a)$ c) $\vec{n} = (a, c)$ d) $\vec{n} = (c, b)$

16. Oxy tekislikda l_1: $a_1x + b_1y + c_1 = 0$ va l_2: $a_2x + b_2y + c_2 = 0$ ikkita to`g`ri chiziq orasidagi burchakni toping.

a) $\left(\widehat{l_1, l_2}\right) = \arccos\frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}$ b) $\left(\widehat{l_1, l_2}\right) = \arccos\frac{a_1a_2-b_1b_2}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}$

c) $\left(\widehat{l_1, l_2}\right) = \arccos\frac{a_1a_2+b_1b_2}{\sqrt{a_1^2-b_1^2}\cdot\sqrt{a_2^2+b_2^2}}$ d) $\left(\widehat{l_1, l_2}\right) = \arccos\frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2-b_2^2}}$.

17. Agar to`g`ri chiziqlar l_1: $y = k_1x + b$ va l_2: $y = k_2x + b$ tenglamalar bilan berilgan bo`lsa, ular orasidagi $\varphi = \left(\widehat{l_1, l_2}\right)$ burchakni toping.

a) $\varphi = arctg\frac{k_2-k_1}{1+k_1k_2}$. b) $\varphi = arctg\frac{k_2-k_1}{1-k_1k_2}$.

c) $\varphi = arctg\frac{k_2+k_1}{1+k_1k_2}$. d) $\varphi = arctg\frac{k_2+k_1}{1-k_1k_2}$.

18. l: $ax + by + c = 0$ to`g`ri chiziq va $P(x_0, y_0)$ nuqta orasidagi masofa toping.

a) $d(P, l) = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$. b) $d(P, l) = \frac{|ax_0-by_0+c|}{\sqrt{a^2+b^2}}$

c) $d(P, l) = \frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$ a) $d(P, l) = \frac{|ax_0-by_0-c|}{\sqrt{a^2+b^2}}$

19. $a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0$ tenglik tekislikdagi ...

a) ikkinchi tartibli algebraik egri chiziqni beradi.

b) birinchi tartibli algebraik chiziqni beradi.

c) uchinchi tartibli algebraik egri chiziqni beradi.

d) to`rtinchi tartibli algebraik egri chiziqni beradi.

20. Tekislikning berilgan ikkita F_1 va F_2 nuqtasigacha bo`lgan masofalari yig`indisi o`zgarmas va $2a$ ga teng bo`lgan nuqtalari to`plami ... deyiladi ($a > 0$).

a) ellips b) giperbola c) parabola d) aylana

21. Tekislikning berilgan ikkita F_1 va F_2 nuqtasigacha bo`lgan masofalari ayirmasining moduli o`zgarmas va $2a$ ga teng bo`lgan nuqtalari to`plami ... deyiladi $(a > 0)$.

a) ellips b) giperbola c) parabola d) aylana

22. Tekislikning berilgan to`g`ri chiziqdan va berilgan nuqtadan bir xil uzoqlikda joylashgan nuqtalari to`plami ... deyiladi.

a) ellips b) giperbola c) parabola d) aylana

23. Ellipsning kanonik tenglamasini ko`rsating.

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

c) $y^2 = 2px$ d) $x^2 + y^2 = r^2$

24. Giperbolaning kanonik tenglamasini ko`rsating.

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ c) $y^2 = 2px$ d) $x^2 + y^2 = r^2$

25. Parabolaning kanonik tenglamasini ko`rsating.

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ c) $y^2 = 2px$ d) $x^2 + y^2 = r^2$

26. Ellips ekstsentritini toping va ekstsentriti uchun qaysi munosabat o`rinli bo`lishini ko`rsating.

a) $e = \frac{c}{a}$ va $0 < e < 1$ b) $e = \frac{c}{a}$ va $e > 1$

c) $e = \frac{a}{c}$ va $e = 1$ d) $e = \frac{c}{a}$ va $0 > e$

27. Giperbola ekstsentritini toping va ekstsentriti uchun qaysi munosabat o`rinli bo`lishini ko`rsating.

a) $e = \frac{c}{a}$ va $0 < e < 1$ b) $e = \frac{c}{a}$ va $e > 1$

c) $e = \frac{a}{c}$ va $e = 1$ d) $e = \frac{c}{a}$ va $0 > e$

28. Giperbolaning asimptotalarni ko`rsating.

a) $y = \pm\frac{b}{a}x.$ b) $y = \pm\frac{a}{c}x.$ c) $y = \pm ax.$ d) $y = \pm bx.$

29. To`g`ri chiziqning burchak koeffitsentli tenglamalari berilgan $y = k_1x + b_1$ va $y = k_2x + b_2$ ularning perpendikulyarlik shartini aniqlang.

a) $k_1 \cdot k_2 = -1$ b) $k_1 \cdot k_2 = 1$ c) $k_1 = k_2 = -1$ d) $k_1 = k_2$

30. Aylananing tenglamasini ko`rsating.

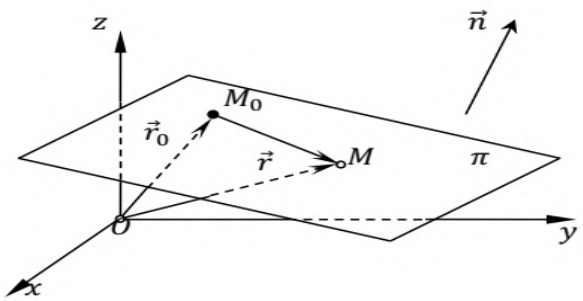
a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ c) $y^2 = 2px$ d) $x^2 + y^2 = r^2$

7-§. FAZODA ANALITIK GEOMETRIYA

7.1. Fazoda tekislik tenglamalari (nazariy qism)

Fazoda tayinlangan $Oxyz$ Dekart koordinatalari tizimini qaraylik. Tekislikning fazodagi bu tizimga nisbatan holatini turli xil usullarda ifodalash mumkin.

π tekislik va undagi *joriy* (yoki *o'zgaruvchi*) *nuqta* deb ataluvchi $M(x, y, z)$ nuqtani qaraymiz. Bu tekislikning biror $M(x_0, y_0, z_0)$ nuqtasini tayinlaylik. $\vec{n}(A, B, C)$ vektor – π tekislikning normali bo'lsin. $\vec{r}_0 = \overrightarrow{OM_0}$, $\vec{r} = \overrightarrow{OM}$ belgilashlar kiritsak, $\overrightarrow{M_0M} = \vec{r} - \vec{r}_0$ tenglikka ega bo'lamiz. $\overrightarrow{M_0M}$ vektor π tekislikda yotgani va $\pi \perp \vec{n}$ ekanligidan $\overrightarrow{M_0M} \perp \vec{n}$ kelib chiqadi. Bu munosabat ushbu tenglikka teng kuchli



$$(\vec{r} - \vec{r}_0)\vec{n} = 0. \qquad (1)$$

Bu tenglik *tekislikning vektor ko'rinishdagi tenglamasi* deyiladi.

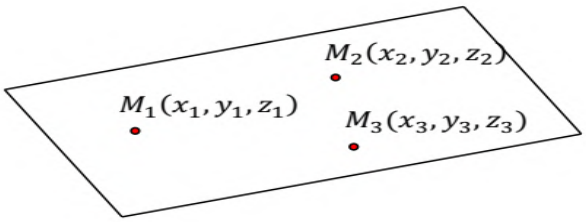
$(\vec{r} - \vec{r}_0)\vec{n}$ skalyar ko'paytmani mos koordinatalar ko'paytmasi yig'indisi ko'rinishda yozib, normal vektori bo'yicha *berilgan nuqtadan o'tuvchi tekislik tenglamasiga* ega bo'lamiz:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \qquad (2)$$

$D = -Ax_0 - By_0 - Cz_0$ belgilash kiritsak, (2) tenglik

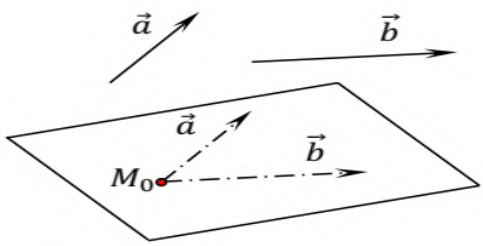
$$Ax + By + Cz + D = 0 \qquad (3)$$

ko'rinishga keladi. (3) tenglik *tekislikning umumiy tenglamasi* deyiladi.



Bitta to'g'ri chiziqda yotmaydigan $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ va $M_3(x_3, y_3, z_3)$ nuqtalar orqali yagona tekislik o'tadi va bu tekislik $(M_1M_2M_3)$ kabi beilgilanadi. $(M_1M_2M_3)$ tekislik tenglamasi ushbu ko'rinishda bo'ladi:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0. \qquad (4)$$



(4) tenglik *berilgan uchta nuqta orqali o'tuvchi tekislik tenglamasi* deyiladi.

Kollinear bo'lmagan $\vec{a}$ va $\vec{b}$ vektorlarni qaraylik. U holda fazoning har bir nuqtasiga shu nuqtadan bu vektorlarga parallel o'tuvchi yagona tekislik mos keladi. Biror M_0 nuqtadan $\vec{a}$ va $\vec{b}$ vektorlarga parallel o'tuvchi tekislik tenglamasi

$$\vec{r} = \vec{r}_0 + t\vec{a} + s\vec{b} \qquad (5)$$

ko‘rinishda bo‘ladi, bu yerda t, s – parametrlar, $-\infty < t < +\infty$, $-\infty < s < +\infty$. (5) tenglama radius vektori $\vec{r}_0 = \overrightarrow{OM_0}$ bo‘lgan M_0 *nuqtadan o‘tuvchi tekislikning parametrik tenglamasi* deb yuritiladi.

7.2. Fazoda tekisliklarning o‘zaro joylashuvi

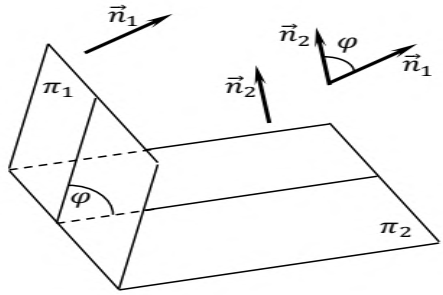
Fazoda ikkita tekislik kesishsa, kesishmada to‘g‘ri chiziq hosil bo‘ladi. bu ikkita tekislik orasidagi burchak ularning normallari orasidagi burchaklardan kichigi orqali topiladi. Agar ikkita tekislik

$$\pi_1: A_1x + B_1y + C_1z + D_1 = 0,$$
$$\pi_2: A_2x + B_2y + C_2z + D_2 = 0,$$

tenglamalar bilan berilgan bo‘lsa, ular orasidagi burchak kosinusi

$$\cos\varphi = \frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}.$$

formula orqali topiladi.



Tenglamalari bilan berilgan ikkita π_1, π_2 tekisliklarning parallel bo‘lishining zaruriy va yetarli shrarti ushbu ko‘rinishda bo‘ladi:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}; \qquad (6)$$

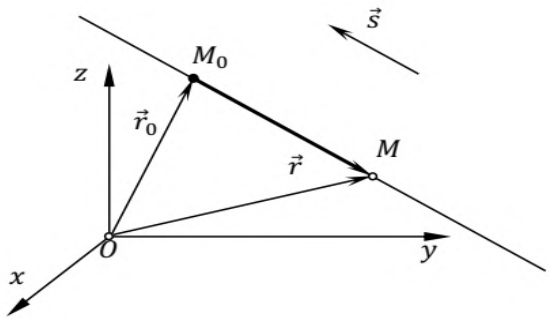
parallel bo‘lishining zaruriy va yetarli shrarti esa ushbu ko‘rinishda bo‘ladi:

$$A_1A_2 + B_1B_2 + C_1C_2 = 0. \qquad (7)$$

7.3. Fazoda to‘g‘ri chiziq tenglamalari

Berilgan $M_0(x_0, y_0, z_0)$ nuqtadan nolnas $\vec{s} = \{m, n, p\}$ vektorga parallel o‘tuvchi l to‘g‘ri chiziq tenglamasi ushbu ko‘rinishda bo‘ladi

$$\vec{r} = \vec{r}_0 + \lambda\vec{s}. \qquad (8)$$



$\vec{s}$ vektor l to‘g‘ri chiziqning *y`onaltiruvchi vektori* deyiladi. Bu yerda $\vec{r}_0 = \overrightarrow{OM_0}$ berilgan M_0 nuqtaning radius-vektori, $\vec{r} = \overrightarrow{OM}$ joriy $M(x, y, z)$ nuqtaning radius-vektori, λ – parametr, $\lambda \in (-\infty, +\infty)$. (8) tenglama fazodagi to‘g‘ri chiziqning *vektor tenglamasi* deyiladi. M_0 nuqta l chiziqning boshlang‘ich nuqtasi deyiladi.

(8) tenglamani ushbu ko‘rinishda

$$x\vec{i} + y\vec{j} + z\vec{k} = (x_0 + \lambda m)\vec{i} + (y_0 + \lambda n)\vec{j} + (z_0 + \lambda p)\vec{k}.$$

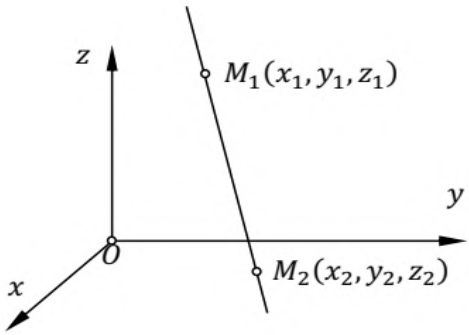
yozib, undan fazodagi to‘g‘ri chiziqning *parametrik tenglamalarini hosil qilamiz*:

$$\begin{cases} x = x_0 + \lambda m, \\ y = y_0 + \lambda n, \\ z = z_0 + \lambda p. \end{cases} \qquad (9)$$

Bu yerda λ – parametr, $\lambda \in (-\infty, +\infty)$.

(9) tenglamada λ parametrni yo‘qotib, fazodagi to‘g‘ri chiziqning ushbu *kanonik tenglamasini hosil qilamiz*:

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}.$$



Ikkita $M_1(x_1,y_1,z_1)$ va $M_2(x_2,y_2,z_2)$ *nuqtalar orqali o‘tuvchi to‘g‘ri chiziq tenglamasi* ushbu ko‘rinishda bo‘ladi:

$$\frac{x-x_0}{x_2-x_1}=\frac{y-y_0}{y_2-y_1}=\frac{z-z_0}{z_2-z_1}.$$

Kesishuvchi ikkita

$$\pi_1: A_1x+B_1y+C_1z+D_1=0,$$
$$\pi_2: A_2x+B_2y+C_2z+D_2=0$$

tekisliklar kesishmasida hosil bo‘lgan to‘g‘ri chiziq ushbu sistema orqali aniqlanadi:

$$\begin{cases}A_1x+B_1y+C_1z+D_1=0,\\ A_2x+B_2y+C_2z+D_2=0.\end{cases}$$

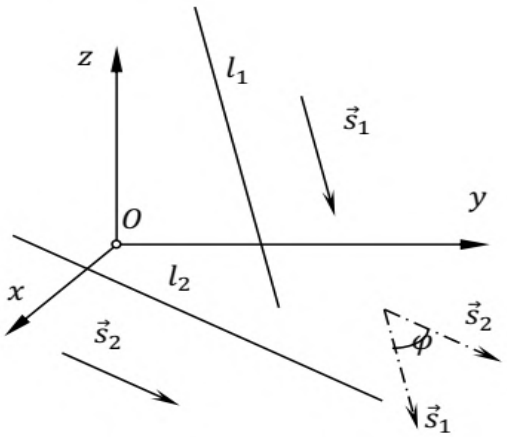
Bu to‘g‘ri chiziq yo‘naltiruvchi vektori ushbu ko‘rinishda bo‘ladi:

$$\vec{s}=\left\{\begin{vmatrix}B_1 & C_1\\ B_2 & C_2\end{vmatrix},-\begin{vmatrix}A_1 & C_1\\ A_2 & C_2\end{vmatrix},\begin{vmatrix}A_1 & B_1\\ A_2 & B_2\end{vmatrix}\right\}.$$

7.4. Fazoda ikkita to‘g‘ri chiziqning o‘zaro joylashuvi

Fazodagi *ikkita to‘g‘ri chiziq orasidagi burchak* deb, ularning yo‘naltiruvchi vektorlari orasidagi burchakning eng kichigiga aytiladi. Agar l_1 va l_2 to‘g‘ri chiziqlar

$$l_1: \frac{x-x_0}{m_1}=\frac{y-y_0}{n_1}=\frac{z-z_0}{p_1} \quad \text{va} \quad l_2: \frac{x-x_0}{m_2}=\frac{y-y_0}{n_2}=\frac{z-z_0}{p_2},$$



tenglamalar bilan berilgan bo‘lsa, ular orasidagi $(l_1\hat{,}l_2)$ burchak kosinusi

$$cos(l_1\hat{,}l_2)=\frac{m_1m_2+n_1n_2+p_1p_2}{\sqrt{m_1^2+n_1^2+p_1^2}\sqrt{m_2^2+n_2^2+p_2^2}} \qquad (10)$$

formula orqali topiladi.

l_1 va l_2 to‘g‘ri chiziqlarning *parallellik* sharti:

$$\frac{m_1}{m_2}=\frac{n_1}{n_2}=\frac{p_1}{p_2}. \qquad (11)$$

l_1 va l_2 to‘g‘ri chiziqlarning *perpendikulyarlik* sharti:

$$m_1m_2+n_1n_2+p_1p_2=0. \qquad (12)$$

Ayqash l_1 va l_2 to‘g‘ri chiziq ushbu vektor tenglamalar bilan berilgan bo‘lsin:

$$l_1: \vec{r}=\vec{r}_1+\lambda\vec{s}_1 \quad \text{va} \quad l_2: \vec{r}=\vec{r}_2+\lambda\vec{s}_2,$$

bunda $\vec{s}_1=\{m_1,n_1,p_1\}$ va $\vec{s}_2=\{m_2,n_2,p_2\}$ – yo‘naltiruvchi vektorlar, $\vec{r}_1=\{x_1,y_1,z_1\}$ va $\vec{r}_2=\{x_2,y_2,z_2\}$ – l_1 va l_2 to‘g‘ri chiziqlarning boshlang‘ich nuqtalarining radius nuqtalari bo‘lsin. U holda ular orasidagi masofa ushbu formula orqali hisoblanishi mumkin:

$$\rho(l_1,l_2)=\frac{\left|det\begin{pmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \end{pmatrix}\right|}{\sqrt{\begin{vmatrix} n_1 & p_1 \\ n_2 & p_2 \end{vmatrix}^2+\begin{vmatrix} m_1 & p_1 \\ m_2 & p_2 \end{vmatrix}^2+\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2}}. \qquad (13)$$

Agar l_1 va l_2 to‘g‘ri chiziqlar parallel bo‘lsa, ular orasidagi masofa

$$\rho(l_1,l_2)=\frac{\sqrt{\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ n_2 & p_2 \end{vmatrix}^2+\begin{vmatrix} x_2-x_1 & z_2-z_1 \\ m_2 & p_2 \end{vmatrix}^2+\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ m_2 & n_2 \end{vmatrix}^2}}{\sqrt{m_2^2+n_2^2+p_2^2}} \text{ yoki}$$

$$\rho(l_1,l_2)=\frac{\sqrt{\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ n_1 & p_1 \end{vmatrix}^2+\begin{vmatrix} x_2-x_1 & z_2-z_1 \\ m_1 & p_1 \end{vmatrix}^2+\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ m_1 & n_1 \end{vmatrix}^2}}{\sqrt{m_1^2+n_1^2+p_1^2}} \qquad (13')$$

formulalardan bittasi orqali hisoblanadi.

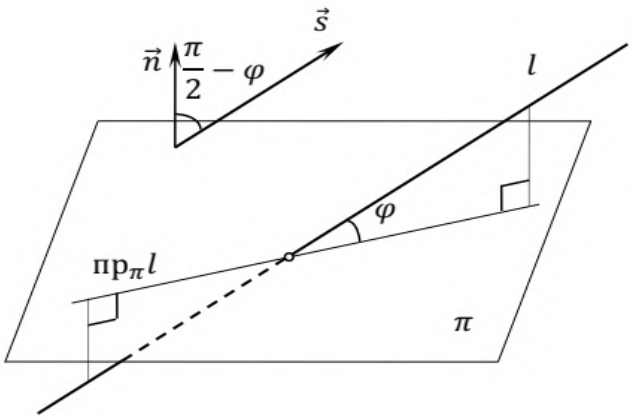
$M(x_1,y_1,z_1)$ nuqta va l: $\vec{r}=\vec{r}_0+\lambda\vec{s}$ to‘g‘ri chiziq orasidagi masofa

$$\rho(M,l)=\frac{\sqrt{\begin{vmatrix} y_1-y_0 & z_1-z_0 \\ n & p \end{vmatrix}^2+\begin{vmatrix} x_1-x_0 & z_1-z_0 \\ m & p \end{vmatrix}^2+\begin{vmatrix} x_1-x_0 & y_1-y_0 \\ m & n \end{vmatrix}^2}}{\sqrt{m^2+n^2+p^2}}.$$

tenglik orqali topiladi. Bu yerda $\vec{r}_0=\{x_0,y_0,z_0\}$ – l to‘g‘ri chiziq boshlang‘ich nuqtasining radius-vektori, $\vec{s}=\{m,n,p\}$ – l to‘g‘ri chiziqning y`onaltiruvchi vektori.

7.5. Fazoda to‘g‘ri chiziq va tekislikning o‘zaro joylashuvi

l to‘g‘ri chiziq va π tekislik orasidagi *burchak* deb, bu to‘g‘ri chiziq va uning π tekislikdagi $pr_\pi l$ proyektsiyasi orasidagi burchakka aytiladi.



π tekislik ushbu

$$Ax+By+Cz+D=0 \qquad (14)$$

umumiy tenglama bilan berilgan bo‘lsin. U holda $\vec{n}=\{A,B,C\}$ vektor uning normali bo‘ladi.

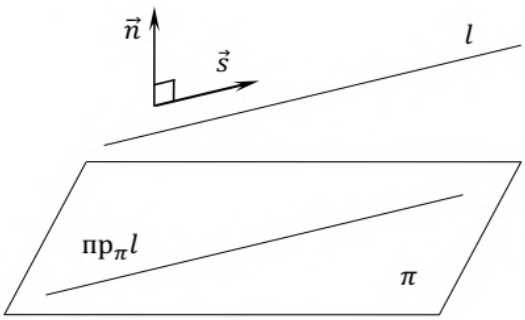
l to‘g‘ri chiziq ushbu

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p} \qquad (15)$$

kanonik tenglama bilan berilgan bo‘lsin. U holda $\vec{s}=\{m,n,p\}$ – uning yo‘naltiruvchi vektori bo‘ladi.

Ravshanki, $(\vec{n},\widehat{\ }\vec{s})=\frac{\pi}{2}-\varphi$. Shuning uchun

$$sin\varphi=\cos\left(\frac{\pi}{2}-\varphi\right)=\frac{mA+nB+pC}{\sqrt{m^2+n^2+p^2}\sqrt{A^2+B^2+C^2}}.$$



l to‘g‘ri chiziq va π tekislikning parallellik ($l||\pi$) sharti

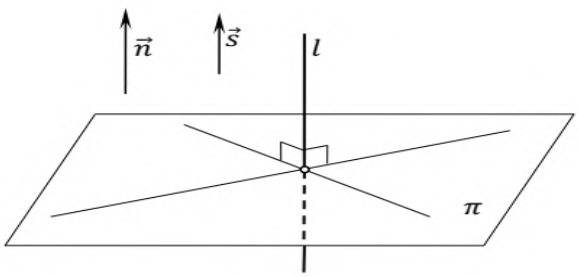
$$mA+nB+pC=0. \qquad (16)$$

l to‘g‘ri chiziq π tekislikdagi l bilan π ning kesishish nuqtasidan o‘tuvchi ixtiyoriy to‘g‘ri chiziqqa perpendikulyar bo‘lsa, l to‘g‘ri chiziq π tekislikka *perpendikulyar* ($l\perp\pi$)

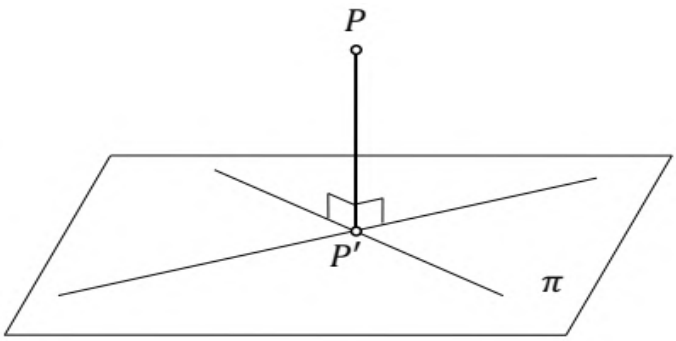
deyiladi.

l to‘g‘ri chiziqning va π tekislikka perpendikulyarlik sharti:

$$\frac{A}{m}=\frac{B}{n}=\frac{C}{p}. \qquad (17)$$



π tekislikning umumiy tenglamasi $Ax+By+Cz+D=0$ bo‘lib, $P(x_p,y_p,z_p)$ – fazoning ixtiyoriy nuqtasi bo‘lsin. P nuqta va π tekislik orasidagi *masofa* deb, PP' kesma uzunligiga aytiladi, bu yerda P' orqali P nuqtaning π tekislikdagi proyektsiyasi belgilangan.



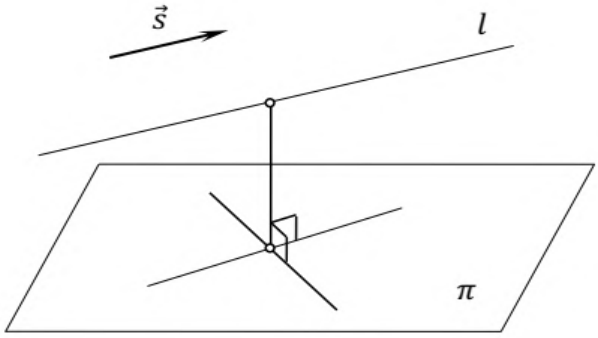
P nuqta va π tekislik orasidagi masofa ushbu formula orqali hisoblanadi:

$$\rho(P,\pi)=\frac{|Ax_p+By_p+Cz_p+D|}{\sqrt{A^2+B^2+C^2}}. \qquad (18)$$

Ushbu $Ax+By+Cz+D=0$ umumiy tenglamasi bilan berilgan π tekislik va $\vec{r}=\vec{r}_0+\lambda\vec{s}$ vektor tenglamasi bilan berilgan l to‘g‘ri chiziq parallel bo‘lsin. U holda l to‘g‘ri chiziq va π tekislik orasidagi masofa

$$\rho(l,\pi)=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}.$$

formula yordamida hisoblanadi.



7.6. Ikkinchi tartibli sirtlarning kanonik tenglamalari. Urinma tekislik

x, y, z o‘zgaruvchilarga nisbatan ikkinchi tartibli algebraik tenglamalar bilan aniqlanadigan nuqtalarning geometrik o‘rni *ikkinchi tartibli sirtlar* deyiladi. Ushbu

$$Ax^2+By^2+Cz^2+2Dxy+2Exz+2Fyz+ax+by+cz+d=0$$

tenglama fazodagi ikkinchi tartibli sirtlarning *umumiy tenglamasi* deyiladi. Bu yerda A, B, C, D, E, F, a, b, c va d – shunday sonlarki, A, B, C, D, E va F lardan kamida bittasi noldan farqli.

Har bir ikkinchi tartibli sirt uchun fazoda shunday $Oxyz$ Dekart koordinatalari sistemasi mavjudki, unda bu sirt kanonik ko‘rinishda yoziladi. Fazoda ikkinchi tartibli sirtlarning 17 tipi mavjud:

1. Mavhum ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$;
2. Haqiqiy ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$;
3. Ikki pallali giperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$;

4. Bir pallali giperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$;

5. Mavhum konus $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$;

6. Haqiqiy konus $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$;

7. Mavhum silindr $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$;

8. Elliptik silindr $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$;

9. Giperbolik silindr $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$;

10. Kesishuvchi mavhum tekisliklar jufti $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$;

11. Kesishuvchi haqiqiy tekisliklar jufti $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$;

12. Parallel mavhum tekisliklar jufti $\frac{x^2}{a^2}=-1$;

13. Parallel haqiqiy tekisliklar jufti $\frac{x^2}{a^2}=1$;

14. Ustma-ust tushuvchi tekisliklar $\frac{x^2}{a^2}=0$;

15. Elliptik paraboloid $z=\frac{x^2}{2p}+\frac{y^2}{2q}$;

16. Giperbolik paraboloid $z=\frac{x^2}{2p}-\frac{y^2}{2q}$;

17. Parabolik paraboloid $z=\frac{x^2}{2p}$.

Agarda to‘g‘ri chiziq sirt bilan ikkita umumiy nuqtaga ega bo‘lsa, bu to‘g‘ri chiziq sirtda to‘laligicha yotadi va u shu sirtning *to‘g‘ri chiziqli yasovchisi* deyiladi.

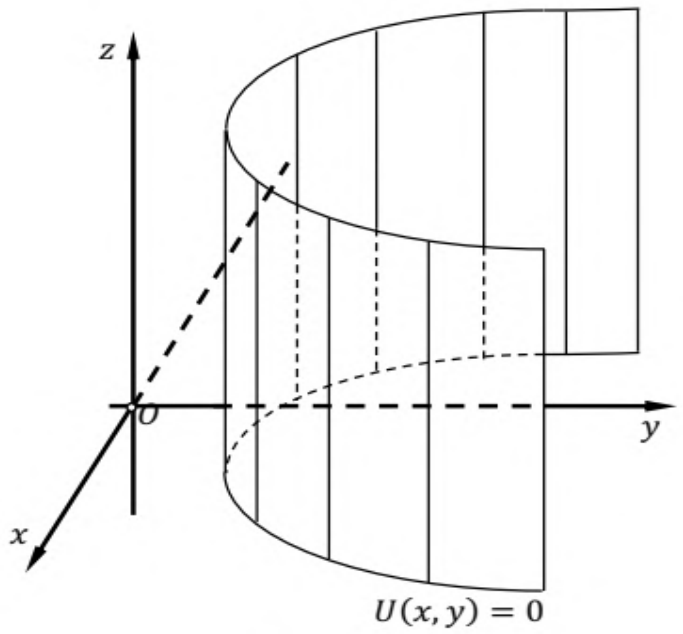
Ikkinchi tartibli sirtlarning quyidagi muhim xossasini keltiramiz: agarda ikkinchi tartibli sirt tekislik bilan kesishsa, u holda kesishmaning nuqtalari to‘plami tartibi ikkidan katta bo‘lmagan chiziq bo‘ladi. Bu holda sirt va tekislik kesishmasi nuqtalari to‘plami *kesim* deb ataladi. Sirtning simmetriya tekisligi bilan kesishishidan hosil bo‘lgan kesim *bosh kesim* deyiladi; bosh kesimlarning uchlari va o‘qlari sirtning *uchlari* va *o‘qlari* deb yuritiladi.

Sirtning berilgan nuqtadagi *urinma tekisligi* deb, sirtga shu nuqtada urinuvchi barcha to‘g‘ri chiziqlar to‘plamiga aytiladi. Sirtning (x_0, y_0, z_0) nuqtadagi urinma tekisligi ushbu jadval orqali topiladi:

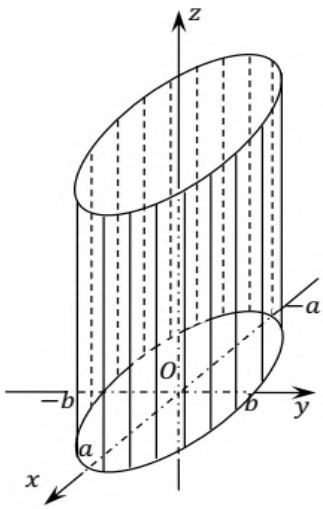
Sirt tenglamasi	Urinma tekislik tenglamasi
$\frac{x^2}{a^2}+\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=\pm1.$	$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}\pm\frac{zz_0}{c^2}=\pm1.$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=0.$	$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}\pm\frac{zz_0}{c^2}=0.$
$z=\frac{x^2}{2p}\pm\frac{y^2}{2q}.$	$z=\frac{xx_0}{2p}\pm\frac{yy_0}{2q}.$

Urinish nuqtasidan urinma tekislikka perpendikulyar o‘tuvchi to‘g‘ri chiziq sirtning shu nuqtadagi *normali* deyiladi.

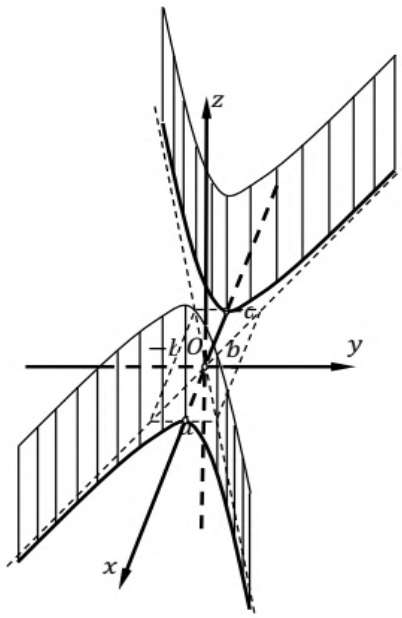
Ayrim ikkinchi tartibli sirtlarning tasvirlari



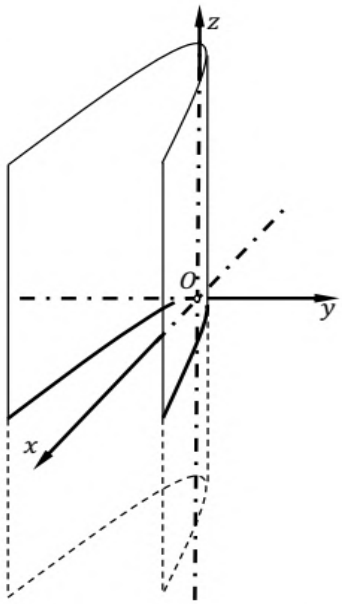
Silindrik sirt



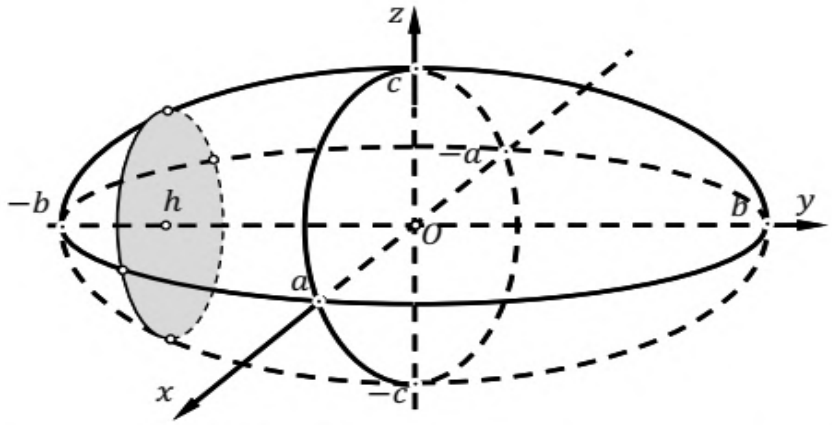
Elliptik silindr
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$



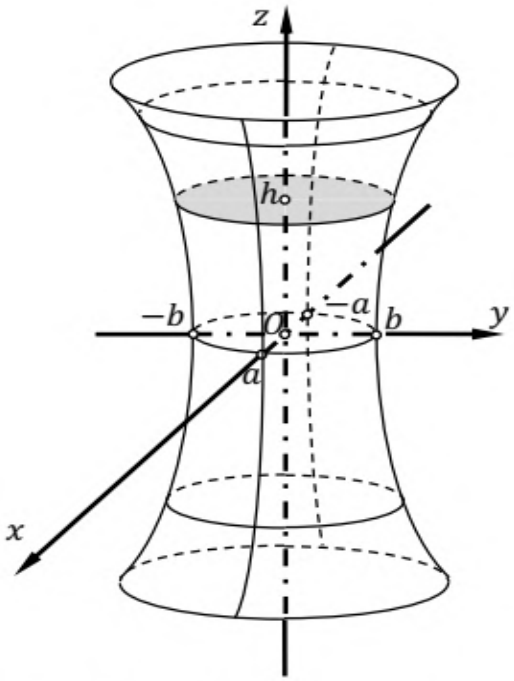
Giperbolik silindr
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$



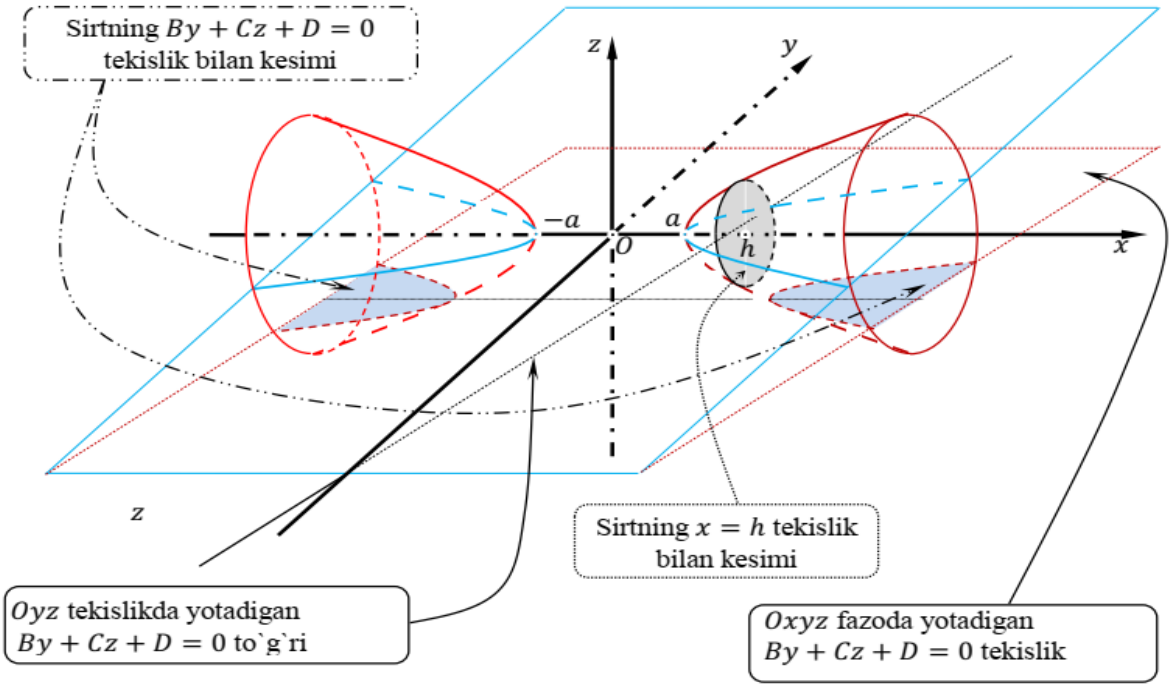
Parabolik silindr
$y^2 = 2px.$



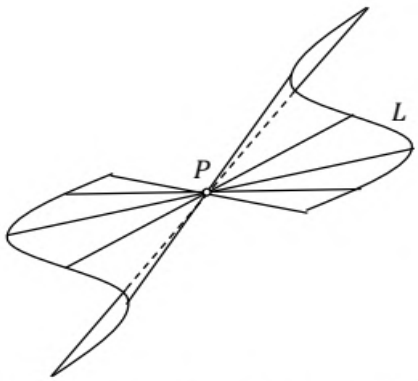
Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $(a>0,\ b>0,\ c>0)$.



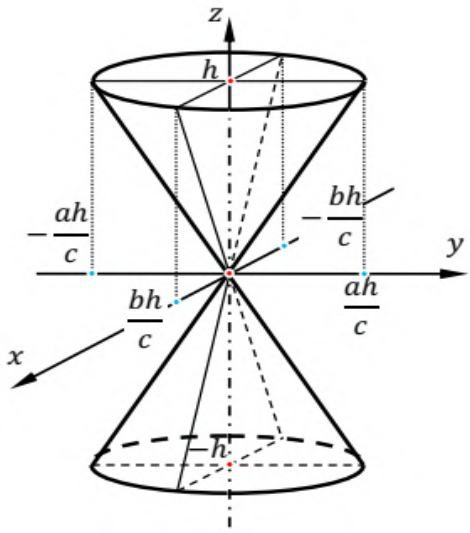
Bir pallali giperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, $a>0$, $b>0$ va $c>0$.

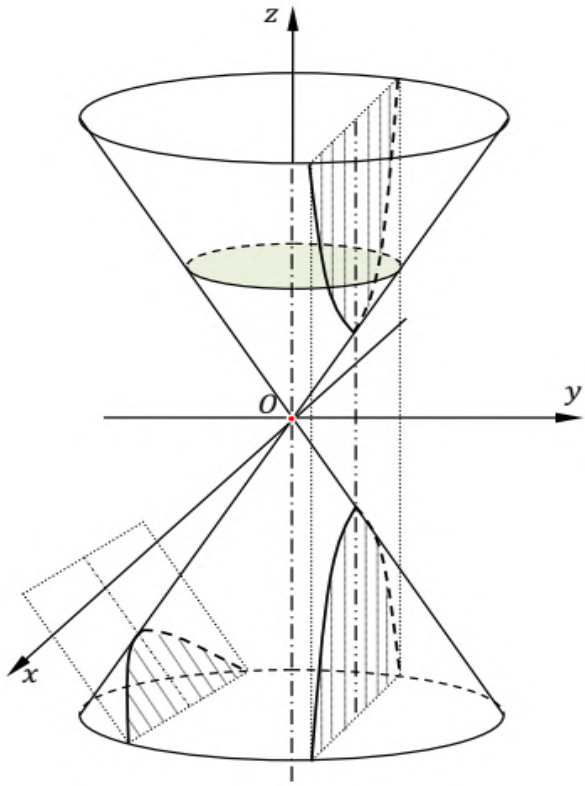


Ikki pallali giperboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $a > 0$, $b > 0$ va $c > 0$.

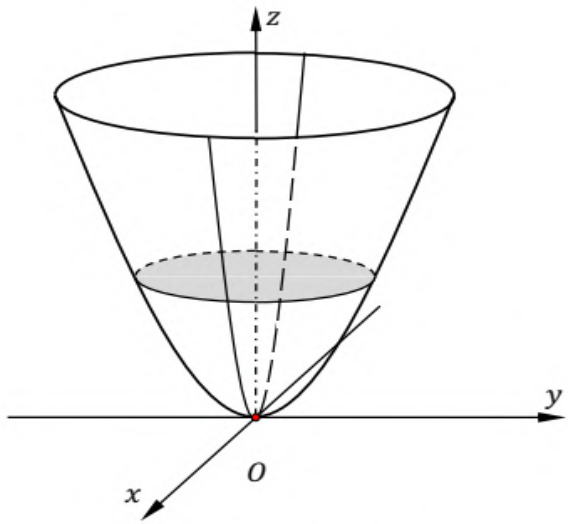


Konik sirt: P – konik sirt uchi, L – konik sirt yo‘naltiruvchisi, P va L egri chiziqlar orqali o‘tuvchi to‘g‘ri chiziqlar – konik sirt yasovchilari.

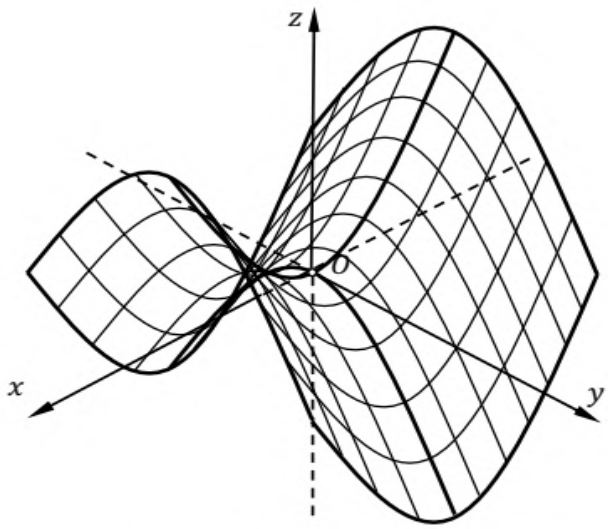




Konus $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$



Elliptic paraboloid
$\frac{x^2}{p} + \frac{y^2}{q} = 2z, p > 0, q > 0.$



Giperbolik paraboloid
$\frac{x^2}{p} - \frac{y^2}{q} = 2z, p > 0, q > 0.$

Mustahkamlash uchun mashqlar

Tekisliklarga oid masalalar

1

1. Quyidagilarga ko`ra tekislik tenglamasini tuzing:

a) O_{xy} tekislikka parallel va $M_0(7;-3;5)$ nuqtadan o`tgan;

b) O_z o`qidan va $M_0(-3;1;-2)$ nuqtadan o`tgan;

c) O_x o`qiga parallel hamda $M_1(4;0;-2)$ va $M_2(5;1;7)$ nuqtalardan o`tgan;

d) $B(2;1;-1)$ nuqtadan o`tgan va $\vec{n}(1,-2,3)$ normal vektoriga;

e) $C(3;4;-5)$ nuqtadan o`tgan hamda $\vec{a}(3,1,-1)$ va $\vec{b}(1,-2,1)$ vektorilarga parallel;

2. $A(5;4;3)$, $B(2;3;-2)$, $C(3;4;2)$, $D(-1;2;1)$ tetraedrning uchlari bo`lsa uning yoqlari tenglamasini tuzing.

3. $x-2y+2z-3=0$ va $3x-4y+5=0$ tekisliklar orasidagi burchakni toping.

4. $2x-y+3z-11=0$ va $2x-y+3z+13=0$ parallel tekisliklar orasidagi masofani toping.

2

1. $A(0;-1;3)$ va $B(1;3;5)$ nuqtalar berilgan, A nuqtadan o‘tuvchi va $\vec{n}=\overrightarrow{AB}$ vektorga perpendikulyar tekislik tenglamasini yozing.

2. $A(a;a;0)$ nuqtadan o‘tuvchi va $\overrightarrow{OA}$ vektorga perpendikulyar tekislik tenglamasini yozing.

3. $A\left(a;-\frac{a}{2};a\right)$ va $B\left(0;\frac{a}{2};0\right)$ nuqtadan teng uzoqlikda bo‘lgan nuqtalar geometrik o‘rnining tenglamasini yozing.

4. $A(0;1;5)$ va $B(2;4;5)$ nuqtalardan o‘tuvchi va Ox o‘qqa parallel tekislik tenglamasini yozing.

5. Ox o‘qdan va $A(0;2;5)$ nuqtadan o‘tuvchi tekislik tenglamasini yozing.

6. Oz o‘qdan va $A(0;2;7)$ nuqtadan o‘tuvchi tekislik tenglamasini yozing.

7. $x-2y+2z-8=0$ va $x+z-6=0$ tekisliklar orasidagi burchakni toping.

8. $x+6z-6=0$ va $x+2y-6=0$ tekisliklar orasidagi burchakni toping.

9. $A(0;1;5)$ $B(5;4;5)$ va $C(2;4;5)$ nuqtalardan o‘tuvchi tekislik tenglamasini yozing.

10. $A(-2;1;-5)$ $B(1;0;2)$ va $C(2;4;5)$ nuqtalardan o‘tuvchi tekislik tenglamasini yozing.

2. Fazoda to‘g‘ri chiziqlarga oid masalalar

1. $M_0(2,0,-3)$ nuqtadan:

а) $\vec{s}(2,-3,5)$ vektorga parallel;

б) $\begin{cases} 2x-y+3z-11=0, \\ 5x+4y-z+8=0. \end{cases}$ to`g`ri chiziqqa parallel;

o`tgan to`g`ri chiziqning kanonik teglamasini tuzing

2. To`g`ri chiziq va tekislikning o`zaro vaziyatini aniqlang, ular kesishsa u holda kesishish nuqtasini toping:

а) $\frac{x+1}{2}=\frac{y-3}{4}=\frac{z}{3}$ va $3x-3y+2z-5=0$;

б) $\frac{x-13}{8}=\frac{y-1}{2}=\frac{z-4}{3}$ va $x+2y-4z+1=0$;

в) $\frac{x-7}{5}=\frac{y-4}{1}=\frac{z-5}{4}$ va $3x-y+2z-5=0$.

3. $\begin{cases} x=2 \\ y=3 \end{cases}$ to‘g‘ri chiziqning yo‘naltiruvchi vektori toping.

4. $B(5;4;5)$ va $C(2;4;5)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamalarini yozing va uning yo‘naltiruvchi vektorini toping.

5. $B(-5;2;-1)$ va $C(-2;-4;5)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamalarini yozing va uning yo‘naltiruvchi vektorini toping.

6. $\frac{x+1}{3}=\frac{y-1}{4}=\frac{z-3}{-2}$ to‘g‘ri chiziq $2x+y+5z-6=0$ tekislikka parallel ekanlini tekshiring.

7. $\begin{cases} 2x+y+8z-2=0 \\ x-2y-z+2=0 \end{cases}$ to‘g‘ri chiziqni a) paremetrik ko‘rinishda b) kanonik ko‘rinishda yozing.

8. $\begin{cases} 2x-y-2=0 \\ 2x--z+2=0 \end{cases}$ va $\begin{cases} 3x-2y+8=0 \\ -3x+z=0 \end{cases}$ to‘g‘ri chiziqlar orasidagi burchakni toping.

O`z-o`zini tekshirish uchun savollar

1. Fazoda tekislik tenglamalarini yozing.
2. Fazoda tekisliklarning o‘zaro joylashuvi qanday aniqlanadi?
3. Fazoda to‘g‘ri chiziq tenglamalarini yozing.
4. Fazoda ikkita to‘g‘ri chiziqning o‘zaro joylashuvi qanday aniqlanadi?
5. Fazoda to‘g‘ri chiziq va tekislikning o‘zaro joylashuvi qanday aniqlanadi?
6. Ikkinchi tartibli sirtlarning kanonik tenglamalarini yozing.
7. Urinma tekislik deb nimaga aytiladi?

O`z-o`zini tekshirish uchun testlar

Fazoda analitik geometriyaga oid testlar

1. $M(x_0,y_0,z_0)$ nuqtadan o`tib, $\vec{n}(A,B,C)$ vektorga perependikulyar tekislikning vektor ko‘rinishdagi tenglamasini ko`rsating.

a) $(\vec{r}-\vec{r}_0)\vec{n}=0$ b) $(\vec{r}+\vec{r}_0)\vec{n}=0$

c) $(\vec{r}+2\vec{r}_0)\vec{n}=0$ d) $(\vec{r}-\vec{n}\vec{r}_0)\vec{r}=0$

2. $M(x_0,y_0,z_0)$ nuqtadan o`tib, $\vec{n}(A,B,C)$ vektorga perependikulyar tekislikning tenglamasini ko`rsating.

a) $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

b) $A(x+x_0)+B(y+y_0)+C(z-z_0)=0$

c) $A(x-x_0)+B(y-y_0)+C(z+z_0)=0$
d) $A(x+x_0)+B(y-y_0)+C(z+z_0)=0$

3. Tekislikning umumiy tenglamasini ko`rsating.

a) $Ax+By+Cz+D=0$ b) $Ax-By+Cz+D=0$
c) $Ax+By-Cz=0$ d) $Ax-By-Cz=0$

4. Bitta to‘g‘ri chiziqda yotmaydigan $M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$ va $M_3(x_3,y_3,z_3)$ nuqtalar orqali o‘tadigan tekislik tenglamasini ko‘rsating.

a) $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$ b) $\begin{vmatrix} x+x_1 & y-y_1 & z-z_1 \\ x_2+x_1 & y_2-y_1 & z_2-z_1 \\ x_3+x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

c) $\begin{vmatrix} x-x_1 & y+y_1 & z-z_1 \\ x_2-x_1 & y_2+y_1 & z_2-z_1 \\ x_3-x_1 & y_3+y_1 & z_3-z_1 \end{vmatrix}=0$ d) $\begin{vmatrix} x-x_1 & y-y_1 & z+z_1 \\ x_2-x_1 & y_2-y_1 & z_2+z_1 \\ x_3-x_1 & y_3-y_1 & z_3+z_1 \end{vmatrix}=0$

5. M_0 nuqtadan $\vec{a}$ va $\vec{b}$ vektorlarga parallel o‘tuvchi tekislik tenglamasini ko‘rsating.

a) $\vec{r}=\vec{r}_0+t\vec{a}+s\vec{b}$ b) $\vec{r}=\vec{r}_0-s\vec{b}$ c) $\vec{r}=-t\vec{a}s\vec{b}$ d) $\vec{r}=\vec{r}_0-t\vec{a}$

6. Agar ikkita tekislik

$$\pi_1: A_1x+B_1y+C_1z+D_1=0,$$
$$\pi_2: A_2x+B_2y+C_2z+D_2=0,$$

tenglamalar bilan berilgan bo‘lsa, ular orasidagi burchak kosinusi toping.

a) $\cos\varphi=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$ b) $\cos\varphi=\frac{A_1A_2-B_1B_2-C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$

c) $\cos\varphi=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2-B_1^2-C_1^2}\sqrt{A_2^2-B_2^2-C_2^2}}$ d) $\cos\varphi=\frac{A_1A_2-B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2-B_2^2-C_2^2}}$

7. Agar ikkita tekislik

$$\pi_1: A_1x+B_1y+C_1z+D_1=0,$$
$$\pi_2: A_2x+B_2y+C_2z+D_2=0,$$

tenglamalar bilan berilgan bo‘lsa, bu tekisliklarning parallel bo‘lishining zaruriy va yetarli shrartini ko`rsating.

a) $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$ b) $\frac{A_1}{A_2}+\frac{B_1}{B_2}=\frac{C_1}{C_2}$ c) $\frac{A_1}{A_2}\neq\frac{B_1}{B_2}=\frac{C_1}{C_2}$ d) $\frac{A_1}{A_2}\neq\frac{B_1}{B_2}\neq\frac{C_1}{C_2}$

8. Agar ikkita tekislik

$$\pi_1: A_1x+B_1y+C_1z+D_1=0,$$
$$\pi_2: A_2x+B_2y+C_2z+D_2=0,$$

tenglamalar bilan berilgan bo‘lsa, bu tekisliklarning perependikulyar bo‘lishining zaruriy va yetarli shrartini ko`rsating.

a) $A_1A_2+B_1B_2+C_1C_2=0.$ b) $A_1A_2-B_1B_2+C_1C_2=0.$
c) $A_1A_2+B_1B_2-C_1C_2=0.$ d) $A_1A_2-B_1-C_1C_2=0.$

9. Berilgan $M_0(x_0,y_0,z_0)$ nuqtadan nolnas $\vec{s}=\{m,n,p\}$ vektorga parallel o‘tuvchi l to‘g‘ri chiziqning vektor ko`rinishidagi tenglamasini ko`rsating.

a) $\vec{r}=\vec{r}_0+\lambda\vec{s}.$ b) $\vec{r}=\vec{r}_0-\vec{s}.$ c) $\vec{r}=\lambda\vec{s}.$ d) $\vec{r}=\vec{r}_0-\lambda\vec{s}.$

10. Fazodagi to‘g‘ri chiziqning parametrik ko`rinishidagi tenglamasini ko`rsating.

a) $\begin{cases} x = x_0 + \lambda m, \\ y = y_0 + \lambda n, \\ z = z_0 + \lambda p. \end{cases}$ b) $\begin{cases} x = x_0 - \lambda m, \\ y = y_0 - \lambda n, \\ z = z_0 + \lambda p. \end{cases}$

c) $\begin{cases} x = x_0 - \lambda m, \\ y = y_0 + \lambda n, \\ z = z_0 + \lambda p. \end{cases}$ d) $\begin{cases} x = x_0 + \lambda m, \\ y = y_0 - \lambda n, \\ z = z_0 - \lambda p. \end{cases}$

11. Fazodagi to‘g‘ri chiziqning kanonik tenglamasini ko`rsating.

a) $\frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$. b) $\frac{x+x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$.

c) $\frac{x-x_0}{m} = \frac{y+y_0}{n} = \frac{z-z_0}{p}$. d) $\frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z+z_0}{p}$.

12. Ikkita $M_1(x_1, y_1, z_1)$ va $M_2(x_2, y_2, z_2)$ nuqtalar orqali o‘tuvchi to‘g‘ri chiziq tenglamasini ko`rsating.

a) $\frac{x-x_0}{x_2-x_1} = \frac{y-y_0}{y_2-y_1} = \frac{z-z_0}{z_2-z_1}$. b) $\frac{x-x_0}{x_2-x_1} = \frac{y-y_0}{y_2-y_1} = \frac{z-z_0}{z_2-z_1}$.

c) $\frac{x-x_0}{x_2-x_1} = \frac{y-y_0}{y_2-y_1} = \frac{z-z_0}{z_2-z_1}$. d) $\frac{x-x_0}{x_2-x_1} = \frac{y-y_0}{y_2-y_1} = \frac{z-z_0}{z_2-z_1}$.

13. Kesishuvchi ikkita

$$\pi_1: A_1x + B_1y + C_1z + D_1 = 0,$$
$$\pi_2: A_2x + B_2y + C_2z + D_2 = 0$$

tekisliklar kesishmasida hosil bo‘lgan to‘g‘ri chiziq tenglamasini ko`rsating.

a) $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$ b) $\frac{x-x_0}{x_2-x_1} = \frac{y-y_0}{y_2-y_1} = \frac{z-z_0}{z_2-z_1}$.

c) $\frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$. d) $\begin{cases} x = x_0 + \lambda m, \\ y = y_0 + \lambda n, \\ z = z_0 + \lambda p. \end{cases}$

14. $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$ to‘g‘ri chiziqning yo‘naltiruvchi vektorini ko‘rsating.

a) $\vec{s} = \left\{ \begin{vmatrix} B_1 C_1 \\ B_2 C_2 \end{vmatrix}, - \begin{vmatrix} A_1 C_1 \\ A_2 C_2 \end{vmatrix}, \begin{vmatrix} A_1 B_1 \\ A_2 B_2 \end{vmatrix} \right\}$. b) $\vec{s} = \{m, n, p\}$

c) $\vec{n}(A, B, C)$ d) $\vec{s} = \{-m, -n, -p\}$

15. Agar l_1 va l_2 to‘g‘ri chiziqlar

$$l_1: \frac{x-x_0}{m_1} = \frac{y-y_0}{n_1} = \frac{z-z_0}{p_1} \quad \text{va} \quad l_2: \frac{x-x_0}{m_2} = \frac{y-y_0}{n_2} = \frac{z-z_0}{p_2},$$

tenglamalar bilan berilgan bo‘lsa, ular orasidagi $(\widehat{l_1, l_2})$ burchak kosinusi toping.

a) $cos(\widehat{l_1, l_2}) = \frac{m_1m_2+n_1n_2+p_1p_2}{\sqrt{m_1^2+n_1^2+p_1^2}\sqrt{m_2^2+n_2^2+p_2^2}}$

b) $cos(\widehat{l_1, l_2}) = \frac{m_1m_2-n_1n_2-p_1p_2}{\sqrt{m_1^2+n_1^2+p_1^2}\sqrt{m_2^2+n_2^2+p_2^2}}$

c) $cos(\widehat{l_1, l_2}) = \frac{m_1m_2-n_1n_2-p_1p_2}{\sqrt{m_1^2-n_1^2-p_1^2}\sqrt{m_2^2+n_2^2+p_2^2}}$

d) $cos(\widehat{l_1, l_2}) = \frac{m_1m_2-n_1n_2-p_1p_2}{\sqrt{m_1^2-n_1^2-p_1^2}\sqrt{m_2^2-n_2^2-p_2^2}}$

16. Agar l_1 va l_2 to‘g‘ri chiziqlar

$$l_1: \frac{x-x_0}{m_1} = \frac{y-y_0}{n_1} = \frac{z-z_0}{p_1} \quad \text{va} \quad l_2: \frac{x-x_0}{m_2} = \frac{y-y_0}{n_2} = \frac{z-z_0}{p_2},$$

tenglamalar bilan berilgan bo‘lsa, u holda l_1 va l_2 to‘g‘ri chiziqlarning parallellik sharti toping.

a) $\frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2}$. b) $\frac{m_1}{m_2} \neq \frac{n_1}{n_2} \neq \frac{p_1}{p_2}$.

c) $m_1m_2 - n_1n_2 - p_1p_2 = 0$. d) $m_1m_2 + n_1n_2 + p_1p_2 = 0$.

17. Agar l_1 va l_2 to‘g‘ri chiziqlar

$$l_1: \frac{x-x_0}{m_1} = \frac{y-y_0}{n_1} = \frac{z-z_0}{p_1} \quad \text{va} \quad l_2: \frac{x-x_0}{m_2} = \frac{y-y_0}{n_2} = \frac{z-z_0}{p_2},$$

tenglamalar bilan berilgan bo‘lsa, u holda l_1 va l_2 to‘g‘ri chiziqlarning perpendikulyarlik sharti toping.

a) $\frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2}$. b) $\frac{m_1}{m_2} \neq \frac{n_1}{n_2} \neq \frac{p_1}{p_2}$.

c) $m_1m_2 - n_1n_2 - p_1p_2 = 0$. d) $m_1m_2 + n_1n_2 + p_1p_2 = 0$.

18. l: $\frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$ to‘g‘ri chiziq va π: $Ax + By + Cz + D = 0$ tekislik orasidagi burchakni toping.

a) $sin\varphi = \cos\left(\frac{\pi}{2} - \varphi\right) = \frac{mA+nB+pC}{\sqrt{m^2+n^2+p^2}\sqrt{A^2+B^2+C^2}}$

b) $sin\varphi = \cos\left(\frac{\pi}{2} - \varphi\right) = \frac{mA+nB+pC}{\sqrt{m^2+n^2-p^2}\sqrt{A^2+B^2-C^2}}$

c) $sin\varphi = \cos\left(\frac{\pi}{2} - \varphi\right) = \frac{mA+nB+pC}{\sqrt{m^2+n^2-p^2}\sqrt{A^2+B^2-C^2}}$

d) $sin\varphi = \cos\left(\frac{\pi}{2} - \varphi\right) = \frac{mA+nB-pC}{\sqrt{m^2+n^2-p^2}\sqrt{A^2+B^2+C^2}}$

19. l: $\frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$ to‘g‘ri chiziq va π: $Ax + By + Cz + D = 0$ tekislikning parallellik ($l||\pi$) shartini toping.

a) $mA + nB + pC = 0$. b) $mA - nB + pC = 0$.

c) $sin\varphi = \cos\left(\frac{\pi}{2} - \varphi\right) = \frac{mA+nB+pC}{\sqrt{m^2+n^2+p^2}\sqrt{A^2+B^2+C^2}}$

d) $sin\varphi = \cos\left(\frac{\pi}{2} - \varphi\right) = \frac{mA+nB+pC}{\sqrt{m^2+n^2-p^2}\sqrt{A^2+B^2-C^2}}$

20. l: $\frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$ to‘g‘ri chiziq va π: $Ax + By + Cz + D = 0$ tekislikning perpendikulyarlik ($l \perp \pi$) shartini toping.

a) $\frac{A}{m} = \frac{B}{n} = \frac{C}{p}$. b) $mA + nB + pC = 0$.

c) $sin\varphi = \cos\left(\frac{\pi}{2} - \varphi\right) = \frac{mA+nB+pC}{\sqrt{m^2+n^2+p^2}\sqrt{A^2+B^2+C^2}}$

d) $mA - nB + pC = 0$.

21. $P(x_p, y_p, z_p)$ nuqta va π: $Ax + By + Cz + D = 0$ tekislik orasidagi masofani topish formulasini ko`rsating.

a) $\rho(P,\pi) = \frac{|Ax_p+By_p+Cz_p+D|}{\sqrt{A^2+B^2+C^2}}$. b) $\rho(P,\pi) = \frac{|Ax_p-By_p+Cz_p+D|}{\sqrt{A^2+B^2-C^2}}$.

c) $\rho(P,\pi) = \frac{|Ax_p-By_p-Cz_p+D|}{\sqrt{A^2+B^2+C^2}}$. d) $\rho(P,\pi) = \frac{|Ax_p+By_p-D|}{\sqrt{A^2+B^2-C^2}}$.

22. Umumiy tenglamasi bilan berilgan π: $Ax + By + Cz + D = 0$ tekislik va vektor tenglamasi bilan berilgan l: $\vec{r} = \vec{r}_0 + \lambda\vec{s}$ to‘g‘ri chiziq parallel bo‘lsa,u holda l to‘g‘ri chiziq va π tekislik orasidagi masofani toping.

a) $\rho(l,\pi) = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$. b) $\rho(l,\pi) = \frac{|Ax_0-By_0+D|}{\sqrt{A^2+B^2+C^2}}$.

c) $\rho(l,\pi) = \frac{|Ax_0+By_0-Cz_0|}{\sqrt{A^2+B^2+C^2}}$. d) $\rho(l,\pi) = \frac{|Ax_0-Cz_0+D|}{\sqrt{A^2+B^2-C^2}}$.

23. Fazodagi ikkinchi tartibli sirtlarning umumiy tenglamasini ko`rsating.

a) $Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + ax + by + cz + d = 0$

b) $Ax^2 - By^2 + 2Dxy + 2Exz + 2Fyz + ax + by + cz + d = 0$

c) $Ax^2 + 2Dxy + 2Exz - 2Fyz + ax + d = 0$

d) $Ax^2 - Cz^2 + 2Dxy + 2Exz + 2Fyz = 0$

24. Ellipsoidning kanonik ko‘rinishdagi tenglamasini ko`rsating.

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

25. Ikki pallali giperboloidning kanonik ko‘rinishdagi tenglamasini ko`rsating.

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

26. Bir pallali giperboloidning kanonik ko‘rinishdagi tenglamasini ko`rsating.

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$;

27. Konusning kanonik ko‘rinishdagi tenglamasini ko`rsating.

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

28. Giperbolik silindrning kanonik ko‘rinishdagi tenglamasini ko`rsating.

a) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

29. Elliptik silindrning kanonik ko‘rinishdagi tenglamasini ko`rsating.

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

30. Elliptik paraboloidning kanonik ko‘rinishdagi tenglamasini ko`rsating.

a) $z = \frac{x^2}{2p} + \frac{y^2}{2q}$ b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ d) $z = \frac{x^2}{2p}$

8-§. Funksiya ta'rifi va ularning tasnifi

8.1. Matematik mantiq elementlari (nazariy qism)

Matematik mantiq o`rganadigan asosiy asosiy ob'yektlar *mulohazalar* va *predikatlar* hisoblanadi. Hukm chiqaruvchi har qanday darak gap *mulohaza* deyiladi. Mulohazalar katta bosma lotin $A$, $B$, $C$ harflari orqali belgilanadi. *Predikat* – bu jism (ob'yekt)larning qandaydir sohasida aniqlangan, qiymati mulohaza bo`lgan (umuman olganda, bir necha o`zgaruvchili) mantiqiy funksiyadir. Bunda o`zgaruvchilar o`rniga ko`rsatilgan sohadan ob'yektlar qo`yilsa, bu funksiya ushbu ikkita "rost" yoki "yolg`on" (1 yoki 0) qiymatlardan faqat bittasini qabul qiluvchi mulohazaga aylanishi talab etiladi.

Predikatlar mantig`ida **umumiylik kvantori** $\forall$ va **mavjudlik kvantori** $\exists$ muhim ahamiyatga ega. $\forall x\ P(x)$ ifoda "har bir $x$ uchun $P(x)$ mulohaza o`rinli" deb o`qilishi mumkin. $\exists x\ P(x)$ ifoda "shunday $x$ topiladiki, uning uchun $P(x)$ mulohaza o`rinli".

Mulohazalar ustida quyidagi amallarni bajarish mumkin: inkor, kon'yunksiya, diz'yunksiya, implikatsiya, ekvivalensiya. Quyida bu amallarni va ularning rostlik jadvallarini keltiramiz.

$\rceil$ – inkor

A mulohaza faqat va faqat yo`lg`on bo`lganda rost bo`ladigan mulohaza A mulohazaning *inkori* deyiladi va $\rceil A$ kabi belgilanadi. O`qilishi: "A emas".

Rostlik jadvali

A	$\rceil A$
1	0
0	1

$\wedge$ – kon'yunksiya

A va B mulohazalarning ikkalasi ham rost bo`lganda rost, boshqa hollarda yolg`on bo`ladigan mulohaza A va B mulohazalarning kon'yunksiya deyiladi va $A \wedge B$ kabi belgilanadi.

O`qilishi: "A va B".

Rostlik jadvali

A	B	$A \wedge B$
1	1	1
1	0	0
0	1	0
0	0	0

$\vee$ – дизъюнкция

A va B mulohazalarning ikkalasi ham yolg`on bo`lganda yolg`on, boshqa hollarda rost bo`ladigan mulohaza A va B mulohazalarning diz'yunksiyasi deyiladi va $A \vee B$ kabi belgilanadi.

O`qilishi: "A yoki B".

Rostlik jadvali

A	B	$A \vee B$
1	1	1
1	0	1
0	1	1
0	0	0

$\Longrightarrow$ – implikatsiya

A mulohaza rost va B mulohaza yolg`on bo`lgandagina yolg`on qolgan hollarda rost bo`lgan mulohaza A mulohazadan B mulohazaning implikatsiyasi deyiladi va $A \Longrightarrow B$ kabi belgilanadi.

O'qilishi: "A dan B kelib chiqadi".

Rostlik jadvali

A	B	$A \Longrightarrow B$
1	1	1
1	0	0
0	1	1
0	0	1

$\Leftrightarrow$ – ekvivalensiya

A va B mulohazalarning ikkalasi ham rost yoki ikkalasi ham yolg`on bo`lganda rost, qolgan hollarda yolg`on mulohazaga A va B mulohazalarning ekvivalensiya deyiladi va $A \Leftrightarrow B$ kabi belgilanadi.

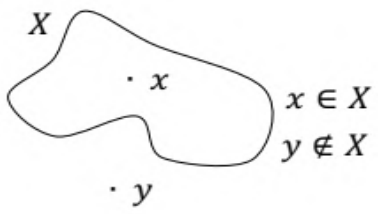
O`qilishi: "A va B teng kuchli".

Rostlik jadvali

A	B	$A \Leftrightarrow B$
1	1	1
1	0	0
0	1	0
0	0	1

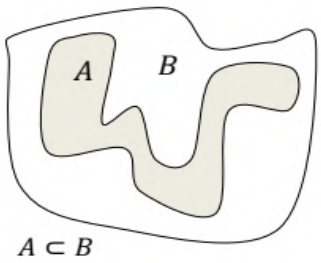
8.2. To`plamlar va ular ustida amallar

To`plamlar nazariyasi matematikaning kata va mustail qismi bo`lib, uning asoslarini bayon etish uchun kamida bir semestrlik kurs talab etilgan bo`lardi. Shuning uchun oliy matematika kursida talab etiladigan eng muhim tushuncha va belgilashlarga to`xtalib o`tamiz. Avvalo t`oplam matematikaning boshlang`ich tushunchasi ekanligini eslatamiz. *Boshlang`ich tushuncha* – bu ta'rifsiz qabul qilinadigan tushunchadir. Boshlang`ich tushuncha *avvaldan* ma'lum deb hisoblanadi va misollar orqali tushuntiriladi. To`plamlar katta bosma lotin A, B, C, … harflari bilan, to`plam elementlari esa kichik yozma lotin a, b, c, … harfliri bilan belgilanadi.



Tegishlilik $\in$ munosabati ham boshlang`ich tushunchadir. $x \in X$ (yoki $X \ni x$) munosabat "$x$ element $X$ to`plamga tegishli", $y \notin X$ (yoki $X \not\ni y$) munosabat esa "y element X to`plamga tegishli emas" deb o`qiladi.

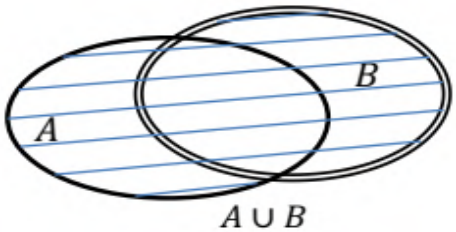
Bironta ham elementga ega bo`lmagan to`plam bo`sh to`plam deyiladi va $\emptyset$ orqali belgilanadi.



A va B to`plamlar aynan bir xil elementlardan tuzilgan bo`lsa, ular **teng** deyiladi va bu holat $A = B$ shaklda yoziladi.

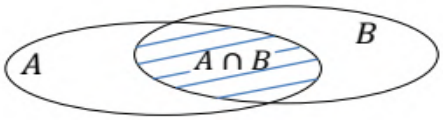
Agar har safar $a \in A$ munosabat $a \in B$ munosabat kelib chiqaversa, u holda A to`plam $B$ to`plamning **qismi** (**to`plamostisi**) deyiladi hamda $A \subset B$ $(B \supset A)$ belgilash ishlatiladi.

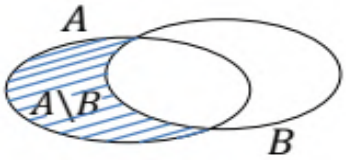
To`plamlar ustida quyidagi amallar bajarilishi mumkin.



$\cup$ – **birlashma.** A va B to`plamlarning birlashmasi deb yo $A$, yo $B$ to`plamlarning kamida bittasiga tegishli elementlardan tuzilgan to`plamga aytiladi. $A$ va $B$ to`plamlarning birlashmasi $A \cup B$ kabi yoziladi.

$\cap$ – **kesishma.** A va B to`plamlarning kesishmasi deb bu to`plamlarning ikkalasiga ham tegishli elementlardan tuzilgan to`plamga aytiladi. $A$ va $B$ to`plamlarning kesishmasi $A \cap B$ kabi yoziladi.





\ – **ayirma.** A to`plamdan $B$ to`plamning ayirmasi deb, A ga tegishli, lekin B ga tegishli bo`lmagan elementlardan iborat to`plamga aytiladi va bu ayirma $A \setminus B$ kabi yoziladi.

8.3. Sonli to`plamlar

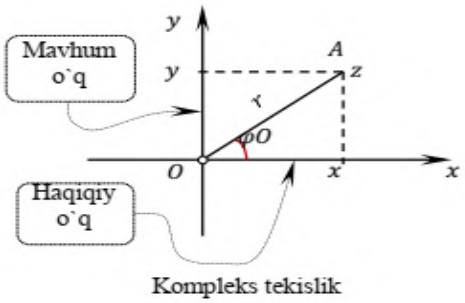
Matematika kursida tez-tez uchrab turadigan to`plamlarni keltiramiz.

1. $\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ – barcha natural sonlar to`plami.
2. $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ – barcha butun sonlar to`plami.
3. $\mathbb{Q} = \{\frac{p}{q}: p \in \mathbb{Z},\ q \in \mathbb{N}\}$ – barcha ratsional sonlar to`plami.
4. $\mathbb{R} = (-\infty;\ +\infty)$ – barcha haqiqiy sonlar to`plami. Har bir haqiqiy $x$ songa $Ox$ o`qining yagona $A(x)$ nuqtasi mos keltiriladi. Shuning uchun bu to`plam ayrim hollarda *son o`qi* deb ham yuritiladi.
5. $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ – irratsional sonlar to`plami.
6. $[a;\ b]$ – kesma (yopiq oraliq). Bu haqiqiy sonlar to`plamining shunday qismiki, uning eng kichigi $a$ soni, eng kattasi $b$ soni bo`lib, bu sonlar kesmaga taalluqli bo`ladi.
7. $(a;\ b)$ – interval (ochiq oraliq). Bu haqiqiy sonlar to`plamining shunday qismiki, u quyidan $a$ soni, yuqoridan $b$ soni bilan chegaralangan bo`lib, bu sonlar intervalga taalluqli bo`lmaydi.
8. $[a;\ b)$ yoki $(a;\ b]$ – yarimintervallar.
9. $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ – o`nli sanoq sistemasining raqamlari.
10. $\mathbb{C} = \{z = x + iy: x \in \mathbb{R},\ y \in \mathbb{R}\}$ – barcha komplekc sonlar to`plami. Bu yerda $i$ – mavhum birlik deyiladi. U shunday kattalikki, $i^2 = -1$ bo`ladi. $x = \operatorname{Re} z$ – kompleks sonning haqiqiy qismi, $y = \operatorname{Im} z$ – mavhum qismi deb yuritiladi.

$z_1 = x_1 + iy_1$ va $z_2 = x_2 + iy_2$ kompleks sonlar uchun $z_1 = z_2 \Leftrightarrow \begin{cases} x_1 = x_2, \\ y_1 = y_2. \end{cases}$

Kompleks sonlar ustida quyidagi **algebraik amallarni** aniqlash mumkin.

Involyutsiya $z = x + iy$ kompleks songa $\bar{z} = x - iy$ kompleks sonni mos qo`yadi. Bunda $z = x + iy$ va $\bar{z} = x - iy$ sonlar *o`zaro qo`shma* sonlar deyiladi. O`zaro qo`shma sonlarning ajoyib xossasi: $z \cdot \bar{z} = x^2 + y^2$.



Algebraik qo`shish: $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$;

Ko`paytirish: $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$;

Bo`lish: $\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_1 y_2 - x_2 y_1}{x_2^2 + y_2^2}$, bu yerda $x_2^2 + y_2^2 \neq 0$.

Har bir $z = x + iy$ kompleks songa Oxy Dekart koordinatalari kiritilgan tekislikning yagona $A(x, y)$ nuqtasi mos keltiriladi. Bu holda Oxy tekislik *kompleks tekislik*, Ox o`q *haqiqiy o`q*, Oy o`q *mavhum o`q* deb yuritiladi. Har bir $z = x + iy$

kompleks son ushbu $z = r(\cos\varphi + isin\,\varphi)$ *trigonometrik shaklda* yozilishi mumkin, bu yerda $r = |z| = \sqrt{x^2 + y^2}$ – berilgan z sonning *moduli*, $\varphi = \arg z = arctg\frac{y}{x}$ – berilgan z sonning *argumenti*.

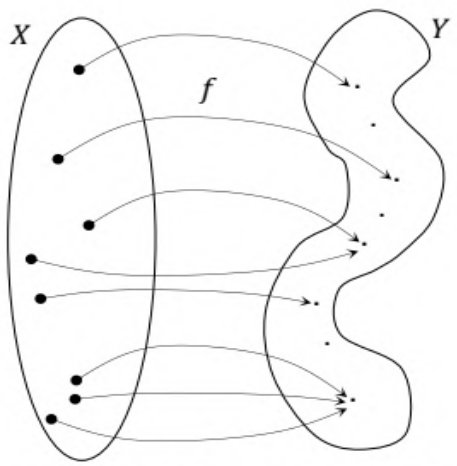
Berilgan $z = x + iy$ sonini n-darajaga ko`tarish, yoki n-darajali ildizdan chiqarish uchun ($n \in \mathbb{N}$) avvalo uni $z = r(\cos\varphi + isin\,\varphi)$ shaklda yoziladi va mos ravishda ushbu formulalardan foydalaniladi:

$$z^n = r^n(\cos n\varphi + i\sin n\varphi), \text{ yoki}$$

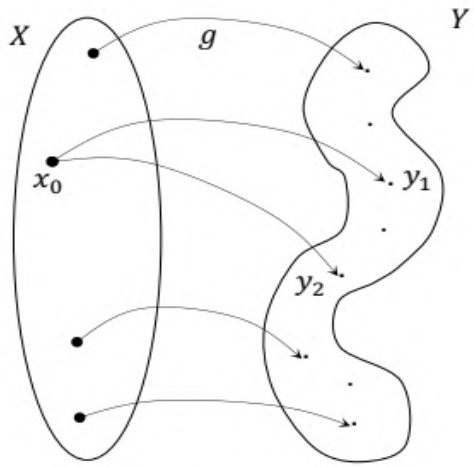
$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi+2\pi k}{n} + i\sin\frac{\varphi+2\pi k}{n}\right), k = 0, ..., n-1.$$

8.4. Funksiya va unung aniqlanish sohasi

X va Y – sonli to`plamlar bo`lsin. Har bir $x \in X$ elementga yagona $y \in Y$ elementni mos qo`yuvchi qonuniyat *funksiya* deyiladi. Funksiyalar f, g, h, φ, ψ , … harflar orqali belgilanadi.

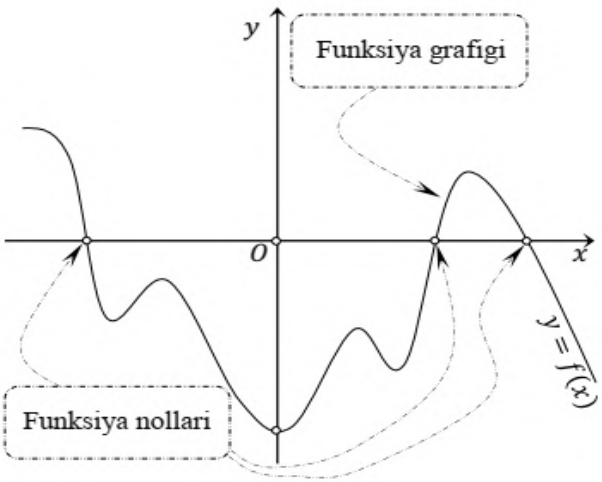


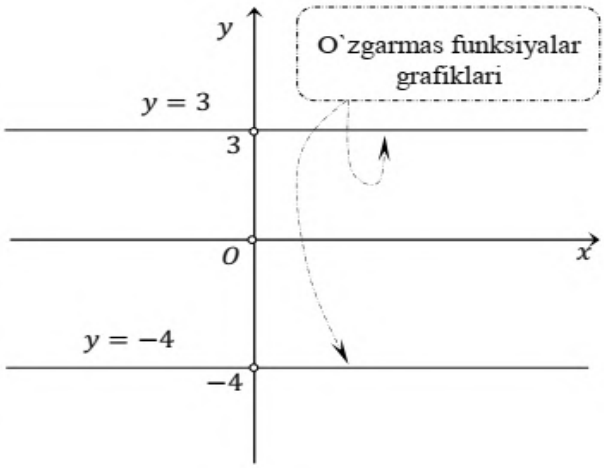
f qonuniyat funksiya bo`ladi. $X$ ning har bir elementiga $Y$ dan yagona $y$ mos qo`yilgan.



g qonuniyat funksiya emas. $x_0 \in X$ elementga ikkita y_1, $y_2 \in Y$ element mos qo`yilgan

$f: X \to Y$ yozuv X to`plamdan $Y$ to`plamga funksiya aniqlanganini bildiradi. $y = f(x)$ yozuv esa f funksiya $x \in X$ elementga $y \in Y$ element mos qo`yilganini anglatadi. Bunda $x$ – *erkli o`zgaruvchi* yoki *argument*, y esa – *erksiz o`zgaruvchi* yoki *funksiya* deb yuritiladi. $x$ argument qabul qilishi mumkin bo`lgan to`plam $y = f(x)$ funksiyaning *aniqlanish sohasi*, $y$ funksiya erishishi mumkin bo`lgan sonlar to`plami $y = f(x)$ funksiyaning *qiymatlar to`plami* deb ataladi. Ular uchun mos ravishda $D(f)$ va $E(f)$ belgilar ishlatiladi.





Eng sodda va muhim funksiyalarda biri bu – *o`zgarmas* funksiyadir. Bu funksiya $X$ dan olingan ixtiyoriy $x$ elementga faqat bitta, aytaylik, $C$ sonni mos qo`yadi. Uni o`zining yagona qiymati C orqali belgilashadi: $C(x) = C, x \in X$.

$x \in D(f)$ bo`lganda $Oxy$ tekislikdagi $(x,\ f(x))$ nuqtalar to`plami $y = f(x)$ funksiyaning *grafigi* deyiladi.

Funksiyalar quyidagi usullarda beriladi:

1. Funksiya berilishining analitik usuli. Agar funksiya bitta yoki bir nechta formulalar yoki tenglamalar bilan berilgan bo`lsa, u holda bu funksiya analitik usulda berilgan deyiladi.

$y = f(x)$ ko`rinishidagi funksiya oshkor berilgan funksiya deyiladi. $y = f(x)$ funksiya $y$ ga nisbatan yechilmagan $F(x,\ y) = 0$ tenlgama bilan berilgan bo`lsa, u holda bu funksiya *oshkormas holda* berilgan deyiladi.

Masalan, $y = \sin x + \cos 2x$ – analitik usulda berilgan oshkor funksiya, $x^2 + y^2 = a^2$ – analitik usulda oshkormas berilgan $y = f(x)$ funksiya bo`ladi.

Ayrim hollarda $y = f(x)$ funksiya $x = \varphi(t)$, $y = \psi(t)$ ko`rinishda t parametr ishtirokida beriladi. Bu funksiyaning *parametrik usulda* berilishidir.

Masalan, $x = a\cos t,\ \ y = b\sin t,\ \ 0 \le t < 2\pi,$ analitik usulda parametrik berilgan $y = f(x)$ funksiya Oxy tekislikda ellipsni aniqlaydi.

2. Funksiya berilishining *jadval usuli* odatda amaliy tajribalarda o`zgaruvchilar orasidagi o`zaro bog`liqlikni o`rnatadi.

3. Ayrim amaliy ishlarda o`zgaruvchilarning bog`liqligi *grafik usulda* beriladi.

$D(f)$ aniqlanish sohasi nolga nisbatan simmetrik $(\forall x \in D(f) \Rightarrow -x \in D(f))$ bo`lgan $y = f(x)$ funksiyani qaralik. Agar $\forall x \in D(f) \Rightarrow f(-x) = f(x)$ bo`lsa, $y = f(x)$ *juft* funksiya, agarda $\forall x \in D(f) \Rightarrow f(-x) = -f(x)$ bo`lsa, $y = f(x)$ *toq* funksiya deyiladi.

$y = f(x)$ – funksiya, $D(f)$ – uning aniqlananish sohasi bo`lsin. Shunday $T > 0$ son topilib, $\forall x \in D(f)$ uchun $(x + T) \in D(f)$ va $f(x + T) = f(x)$ bo`lsa, $y = f(x)$ *davriy* funksiya, T esa $y = f(x)$ funksiyaning *davri* deyiladi.

8.4.1.-misol. $y = \sqrt{4 - x^2} + \frac{1}{x}$ funksiyaning aniqlanish sohasini toping.

Yechish. Berilgan funksiya $4 - x^2 \geq 0$ va $x \neq 0$ bo`lsa, aniqlangan bo`ladi. Ushbu

$$\begin{cases} (2-x)(2+x) \geq 0, \\ x \neq 0, \end{cases}$$

sistemani yechib, bu funksiya argumentning faqatgina $x \in [-2,\ 0) \cup (0,\ 2]$ qiymatlarida ma'noga ega bo`lishini ko`rish mumkin. Shunday qilib,

$$D = [-2,\ 0) \cup (0,\ 2].$$

Bu funksiyaning aniqlanish sohasi simmetrik, lekin o`zi toq ham emas, juft ham emas. Bu funksiya davriy emas.

8.5. Asosiy elementar funksiyalar

Quyidagi funksiyalar *asosiy elementar funksiyalar* deyiladi:

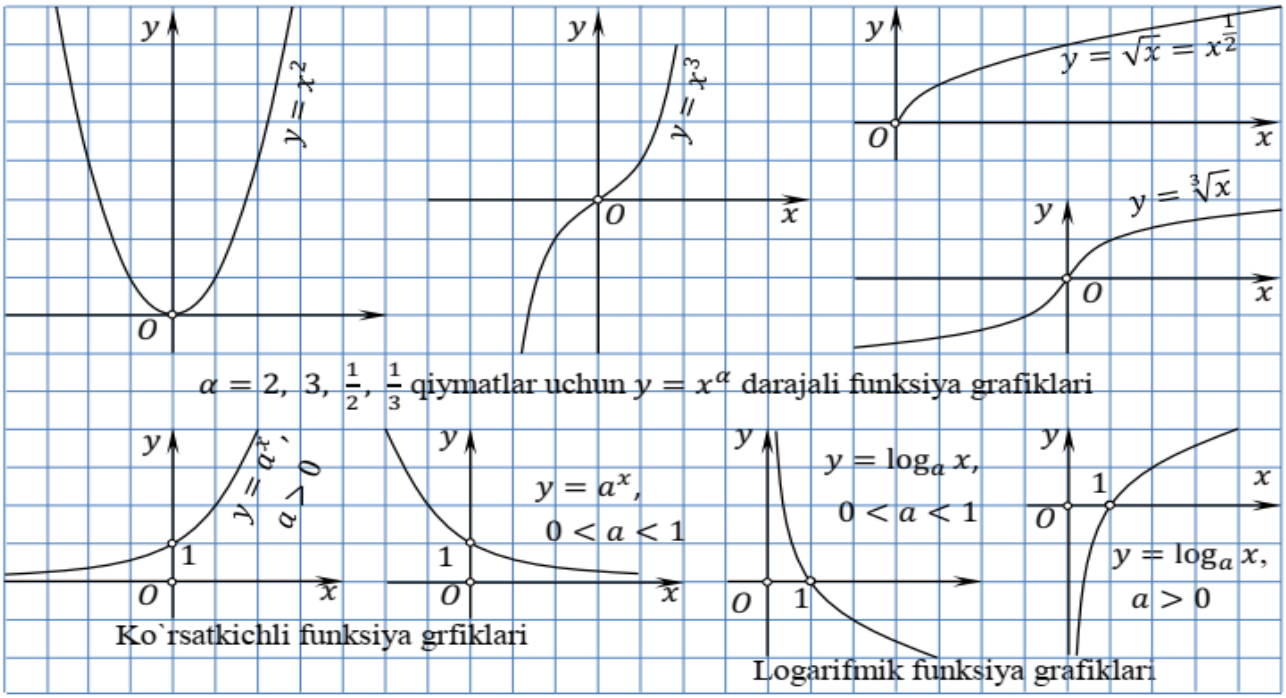
1) *darajali funksiya* $y = x^{\alpha}$, bu yerda α – haqiqiy son;
2) *ko`rsatkichli funksiya* $y = a^x$, bunda a – birdan farqli ixtiyoriy musbat son;
3) *logarifmik funksiya* $y = \log_a x$, bu yerda $a \in \mathbb{R}$, $a > 0$, $a \neq 1$;
4) *trigonometrik funksiyalar*:

$$y = \sin x,\ y = \cos x,\ y = tgx,\ \ y = ctgx;$$

5) *teskari trigonometrik funksiyalar*:

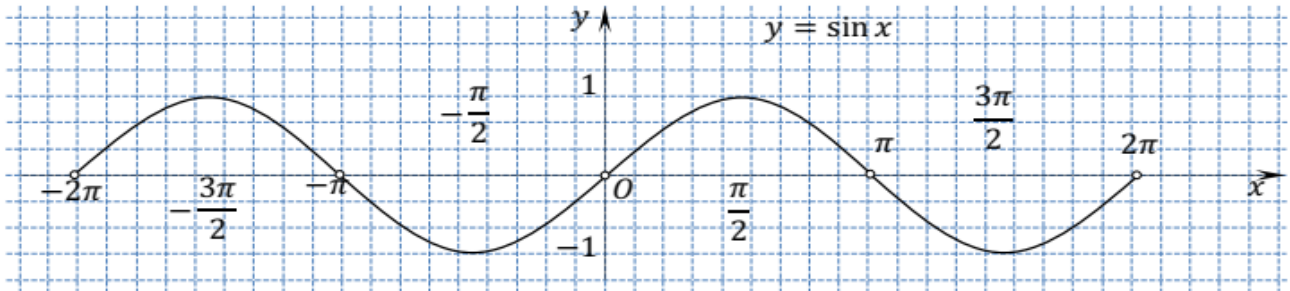
$$y = \arcsin x,\ y = \arccos x,\ y = arctgx,\ \ y = arcctgx.$$

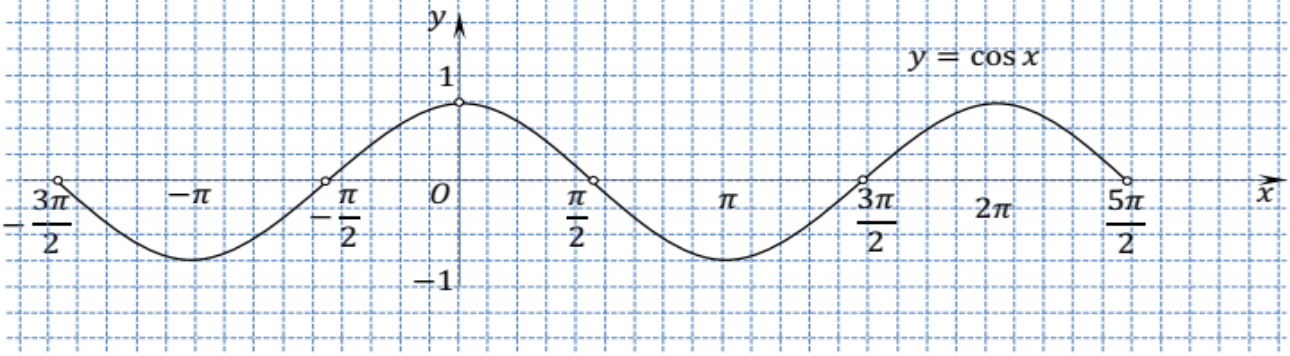
Quyida keltirilgan darajali funksiya grafiklaridan, darajali funksiya uchun aniqlanish sohasi, qiymatlar to`plami ko`rsatkichga bog`liq ekanligi kelib chiqadi.

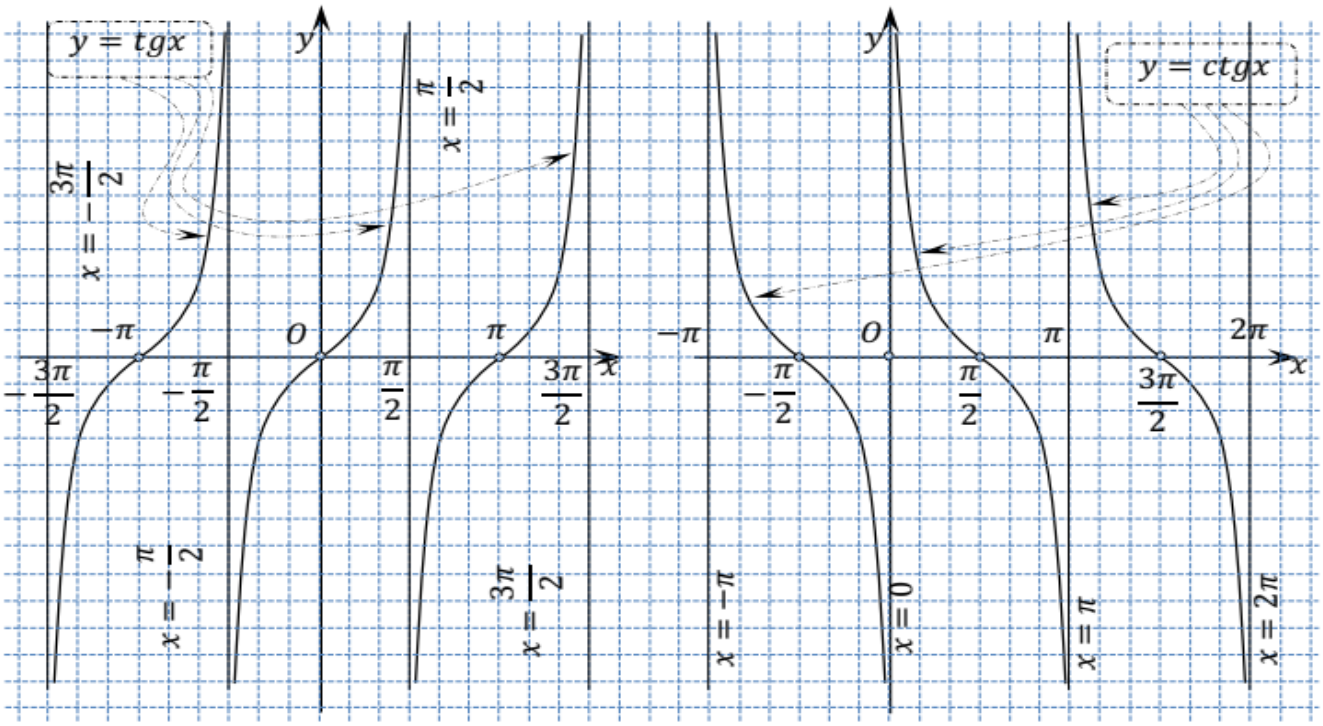


Quyidagilar ko`rinib turibdi: $a > 0$, $a \neq 1$ bo`ladigan a ning qiymatiga bog`liq bo`lmagan holda $D(a^x) = (-\infty, +\infty)$, $E(a^x) = (0, +\infty)$ va $D(\log_a x) = (0, +\infty)$, $E(\log_a x) = (-\infty, +\infty)$ bo`ladi.

Quyidagi rasmlarda trigonometrik funksiyalar grafiklari keltirilgan.







Ravshanki,

$D(\sin x) = (-\infty, +\infty)$, $E(\sin x) = [-1,\ 1]$,

$D(\cos x) = (-\infty, +\infty)$, $E(\cos x) = [-1,\ 1]$,

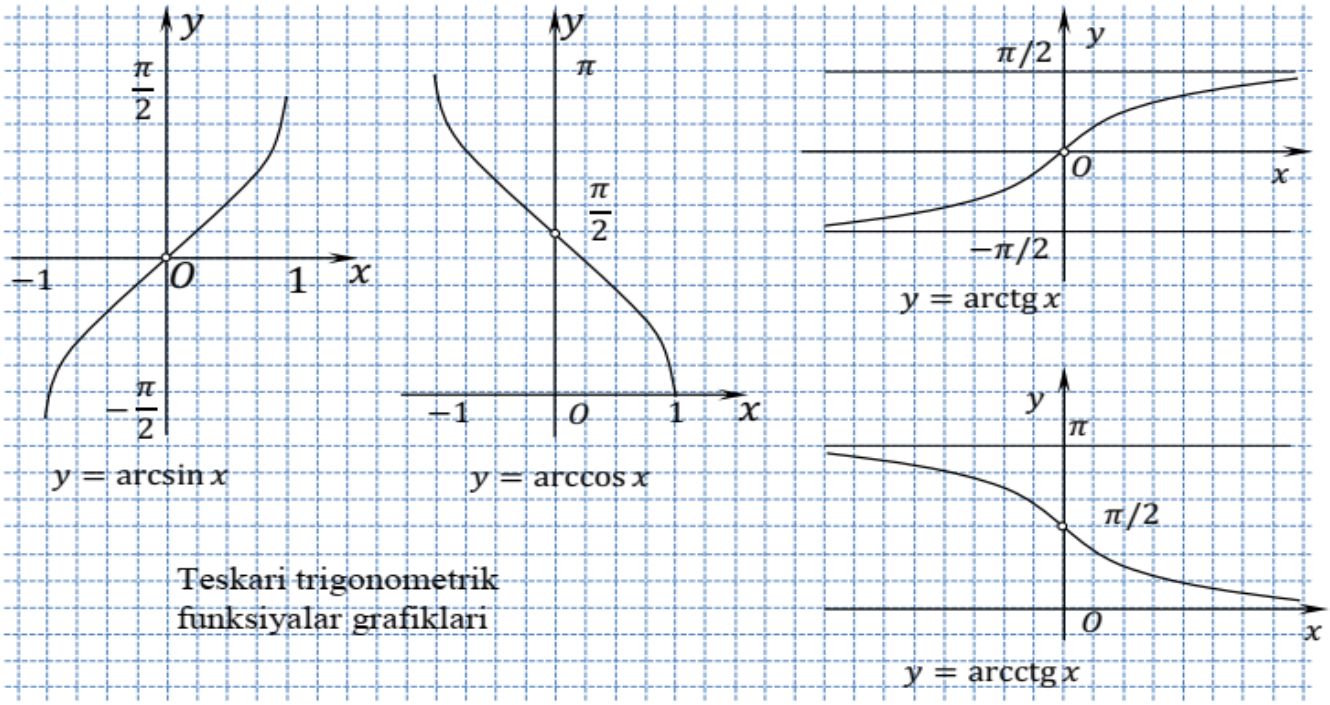
$D(\operatorname{tg} x) = \ldots \cup \left(-\frac{\pi}{2} - 2\pi,\ \frac{\pi}{2} - 2\pi\right) \cup \left(-\frac{\pi}{2} - \pi,\ \frac{\pi}{2} - \pi\right) \cup \left(-\frac{\pi}{2},\ \frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2} + \pi,\ \frac{\pi}{2} + \pi\right) \cup \left(-\frac{\pi}{2} + 2\pi,\ \frac{\pi}{2} + 2\pi\right) \cup \ldots =$

$$= \cup_{k=-\infty}^{+\infty} \left(-\frac{\pi}{2} + k\pi,\ \frac{\pi}{2} + k\pi\right),$$

$E(\operatorname{tg} x) = (-\infty, +\infty)$,

$D(\operatorname{ctg} x) = \cup_{k=-\infty}^{+\infty} (k\pi,\ (k+1)\pi)$, $E(\operatorname{ctg} x) = (-\infty, +\infty)$ bo`ladi.

Teskari trigonometrik funsiyalar grafiklari quyidagicha bo`ladi:



Ko`rinib turibdiki,

$D(arcsin\ x) = [-1,\ \ 1]$, $\qquad E(arcsin\ x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$D(arccos\ x) = [-1,\ \ 1]$, $\qquad E(arccos\ x) = [0, \pi]$,

$D(arctgx) = (-\infty, +\infty)$, $\qquad E(arctg\ x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$D(arcctg\ x) = (-\infty, +\infty)$, $\qquad E(arcctg\ x) = [0, \pi]$

bo`ladi.

Asosiy elementar funksiyalar ustida qo`shish $(+)$, ayirish $(-)$, ko`paytirish $(\times)$, bo`lish $(\div)$ arifmetik amallarini chekli marta qo`llash orqali hosil qilingan funksiyalar *elementar funksiyalar* deyiladi.

Elementar bo`lmaydigan funksiyalar misollarini keltiramiz:

8.5.1.-misol. $y = |x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0 \end{cases}$ – haqiqiy x sonining moduli.

Modulning asosiy xossalari:

1°. $|x| \geq 0$, $\forall x \in \mathbb{R}$; bunda $|x| = 0 \Leftrightarrow x = 0$.

2°. $|x + y| \leq |x| + |y|$, $\forall x,\ y \in \mathbb{R}$;

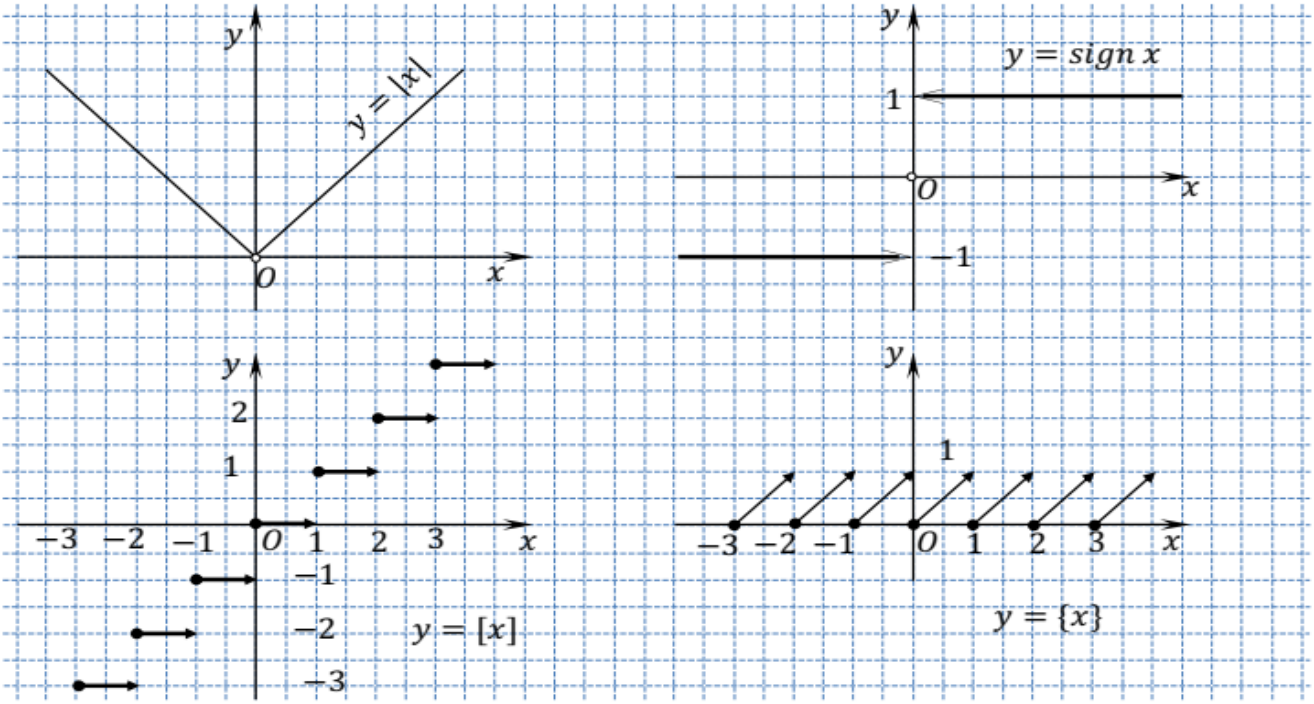
3°. $|x - y| \leq |x| + |y|$, $\forall x,\ y \in \mathbb{R}$;

4°. $|x - y| \geq \big||x| - |y|\big|$, $\forall x,\ y \in \mathbb{R}$;

5°. $|xy| = |x||y|$, $\forall x,\ y \in \mathbb{R}$;

6°. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $\forall x,\ y \in \mathbb{R}$, $y \neq 0$.

8.5.2.-misol. $f(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$ funksiya $sign(x)$ orqali belgilanadi va x sonining ishorasini aniqlaydi. Ravshanki, $D(sign) = (-\infty, +\infty)$, $E(sign) = \{-1,\ 0,\ 1\}$.



8.5.3.-misol. $y = [x]$, bu yerda $[x]$ belgi x haqiqiy sonning butun qismini anglatadi. Uning uchun $D([\cdot]) = (-\infty, +\infty)$, $E([\cdot]) = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ bo`ladi.

8.5.4.-misol. $y = \{x\}$, bu yerda $\{x\}$ yozuv x haqiqiy sonning kasr qismini anglatadi. Uning uchun $D([\cdot]) = (-\infty, +\infty)$, $E([\cdot]) = [0,\ 1)$ bo`ladi.

Musbat x sonlar uchun ushbuga egamiz

$$\{x\} = x - [x].$$

Mnfiy x sonlar uchun kasr qism quyidagicha topiladi

$$\{x\} = 1 - \{|x|\}.$$

8.5.5.-misol. Dirixle funksiasi $f(x) = \begin{cases} 0, & x - \text{irratsional son}, \\ 1, & x - \text{ratsional son}. \end{cases}$

Bu funksiya uchun $D = (-\infty;\ +\infty)$, $E = \{0,\ 1\}$ bo`ladi.

8.5.6.-misol. Dastlabki n ta natural sonning ko`paytmasi $n!$ Orqali belgilanadi va “en-faktorial” deb o`qiladi. Aniqlanishiga ko`ra, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$, va hakozo.

n-faktorialning $n = 0$ va $n = 1$ hollardagi qiymatlari

$$\boxed{\mathbf{0! = 1}} \quad \text{va} \quad \boxed{\mathbf{1! = 1}}$$

tengliklar orqali topiladi. Bu funksiya uchun ushbuga egamiz

$$D = \{0, 1,\ 2,\ \dots,\ n,\ \dots\},\ E = \{1,\ 2,\ 6,\ \dots,\ n!,\ \dots\} \subset \mathbb{N}.$$

8.7. Funksiyalarning tasniflanishi

Elementar funksiyalar sinflari va ularga oid misollarni quyidagi jadvalda keltiramiz:

Algebraik funksiyalar			Transsendent funksiyalar		
Irratsional	Ratsional		Trigonometrik	Teskari trigonometrik	Logarifmik
	Butun	Kasr			
$y=\sqrt[3]{x^2-1}$	$y=\frac{1}{\sqrt{3}}x^4+1$	$y=\frac{2x+3}{3x^4-1}$	$y=\sin x$	$y=arctg\ x$	$y=\ln(x^2-1)$

Elementar funksiyalarning tasniflanishini bilish funksiyalarni integrallashda talab etiladi.

8.8. Murakkab funksiya

Amaliy masalalarni yechishda elementar funksiyalardan tashqari, ancha konstruktiv funkdiyalar ham qaraladi. Masalan, elementar funksiyalarni ketma-ket qo`llash natijasida o`zgaruchilarning yangi bog`lanishlari hosil bo`ladi. Agar X to`plamda $y=f(x)$ funksiya berilgan bo`lib, x argument ham T to`plamda aniqlangan $x=\varphi(t)$ funksiya bo`lsa, u holda T to`plamda $y=f\big(\varphi(t)\big)$ *murakkab funksiya* aniqlangan deyiladi.

8.9. Teskari funksiya

Agarda $y=f(x)$ tenglama x ga nisbatan bir qiymatli yechilsa, ya'ni shunday $x=g(y)$ funksiya topilib, $y=f\big(g(y)\big)$ bajarilsa, u holda $x=g(y)$, yoki odatdagi belgilashlarga ko`ra $y=g(x)$ funksiya berilgan $y=f(x)$ funksiyaga *teskari funksiya* deyiladi.

Funksiya ta'rifi va ularning tasnifiga oid masalalar

1. Funksiyaning aniqlanish sohasini toping.

1) $f(x)=\frac{3x+1}{x^2-1}$

2) $f(x)=\sqrt{5-3x}$

3) $f(x)=\ln(x+2)$

4) $f(x)=2^{\frac{1}{x}}+\arcsin\frac{x+2}{3}$

5) $f(x)=\frac{5}{\sqrt[3]{2x-x^2}}-7\cos 2x$

2. $f^\circ g$ va $g^\circ f$ murakkab funksiyalarni toping.

1)$f(x)=\sqrt{x}$, $\quad g(x)=x^2$

2)$f(x)=x^3$, $\quad g(x)=2x-1$

3)$f(x)=e^x$, $\quad g(x)=\ln x$

4)$f(x)=3x+1$, $\quad g(x)=\cos x$

3. Berilgan funksiyalarga teskari funksiyani toping.

1)$f(x)=x+1$

2)$f(x) = \frac{2}{x+3}$
3)$f(x) = \sqrt{x}$
4)$f(x) = \sqrt{x} + 1$

4. Funksiya grafigini chizing.

1) $y = |x - 3|$
2) $y = x^2 - 6x + 11$
3) $y = 3\cos 2x$
4) $y = -\frac{2}{x} + 1$
5) $y = 2^{x-1} + 3$
6) $y = log_3^{(-x)}$

5. Quyidagi funksiyalarning qaysilari davriy funksiya? Agar davriy funksiya bo`lsa eng kichik davrini toping.

1) $f(x) = \cos\frac{x}{4}$
2) $f(x) = |x|$
3) $f(x) = tg(2x - 1)$
4)$f(x) = \sin\frac{x}{2} - ctgx$
5) $f(x) = \sin 3x \cdot \cos 3x$

6. Quyidagi funksiyalarning qaysilari juft funksiya, qaysilari toq funksiya?

1) $f(x) = \frac{\sin x}{x}$
2) $f(x) = x^5 + 3x^3 - x$
3) $f(x) = \sqrt{x}$
4) $f(x) = \arcsin x$
5) $f(x) = \sin x - ctgx$
6) $f(x) = \sin x - \cos x$
7) $f(x) = |x| - 2$
8) $f(x) = \frac{5}{x^2+2}$

7. $f(x) = x^3 \cdot 2^x$ funksiya uchun quyidgilarni toping.

1) $f(1)$ 2) $f(-3)$ 3) $f(-\sqrt[3]{5})$
4) $f(-x)$ 5) $f(3x)$ 6) $f\left(\frac{1}{x}\right)$

O`z-o`zini tekshirish uchun savollar

1. Matematik mantiq elementlariga nimalar kiradi?
2. Mulohazalar deb nimaga aytiladi?
3. Predikatlar deb nimaga aytiladi?
4. Mulohazalar ustida qanday amallar bajariladi?
5. To`plamlar va ular ustida qanday amallar bajariladi?
6. Funksiya deb nimaga aytiladi?
7. Funksiyning aniqlanish sohasi deb nimaga aytiladi?
8. Funksiyning qiymatlash sohasi deb nimaga aytiladi?

9. Asosiy elementar funksiyalarga qaysi funksiyalar kiradi?
10. Murakkab funksiya deb qanday funksiyaga aytiladi?
11. Teskari funksiya qanday topiladi?

O`z-o`zini tekshirish uchun testlar

Funksiya ta'rifi va ularning tasnifiga oid test

1. Matematik mantiq o`rganadigan asosiy asosiy ob'yektlar … hisoblanadi.
a) mulohazalar va predikatlar b) kvantorlar
c) kon'yunksiya d) mulohaza
2. Hukm chiqaruvchi har qanday darak gap … deyiladi.
a) mulohazalar va predikatlar b) kvantorlar
c) kon'yunksiya d) mulohaza
3. *Predikat* – bu jism (ob'yekt)larning qandaydir sohasida aniqlangan, qiymati mulohaza bo`lgan (umuman olganda, bir necha o`zgaruvchili) ….
a) mantiqiy funksiyadir b) kon'yunksiya
c) diz'yunksiya d) inkor
4. A mulohaza faqat va faqat yo`lg`on bo`lganda rost bo`ladigan mulohaza A mulohazaning … deyiladi.
a) mantiqiy funksiyadir b) kon'yunksiya
c) diz'yunksiya d) inkor
5. A va B mulohazalarning ikkalasi ham rost bo`lganda rost, boshqa hollarda yolg`on bo`ladigan mulohaza A va B mulohazalarning … deyiladi.
a) mantiqiy funksiyadir b) kon'yunksiya
c) diz'yunksiya d) inkor
6. A va B mulohazalarning ikkalasi ham yolg`on bo`lganda yolg`on, boshqa hollarda rost bo`ladigan mulohaza A va B mulohazalarning … deyiladi.
a) mantiqiy funksiyadir b) kon'yunksiya
c) diz'yunksiya d) inkor
7. A mulohaza rost va B mulohaza yolg`on bo`lgandagina yolg`on qolgan hollarda rost bo`lgan mulohaza A mulohazadan B mulohazaning … deyiladi
a) implikatsiyasi b) kon'yunksiya
c) diz'yunksiya d) inkor
8. A va B mulohazalarning ikkalasi ham rost yoki ikkalasi ham yolg`on bo`lganda rost, qolgan hollarda yolg`on mulohazaga A va B mulohazalarning … deyiladi.
a) implikatsiyasi b) kon'yunksiya
c) diz'yunksiya d) ekvivalensiya
9. Bironta ham elementga ega bo`lmagan to`plam … deyiladi va … orqali belgilanadi.
a) bo`sh to`plam, va $\emptyset$ b) birlik to`plam, va E
c) birlik to`plam, va $\emptyset$ d) teng to`plam, va $A = B$

10. A va B to`plamlar aynan bir xil elementlardan tuzilgan bo`lsa, ular deyiladi va bu holat $A = B$ shaklda yoziladi.

a) teng to`plam, va $A = B$ b) birlik to`plam, va E
c) teng to`plam, va $\emptyset$ d) bo`sh to`plam, va E

11. Agar har safar $a \in A$ munosabatdan $a \in B$ munosabat kelib chiqaversa, u holda A to`plam $B$ to`plamning … deyiladi hamda … belgilash ishlatiladi.

a) teng to`plam, va $A = B$ b) birlik to`plam, va E
c) bo`sh to`plam, va $\emptyset$ d) qism to`plami, va $A \subset B$

12. A va B to`plamlarning … deb yo $A$, yo $B$ to`plamlarning kamida bittasiga tegishli elementlardan tuzilgan to`plamga aytiladi. $A$ va $B$ to`plamlarning … kabi yoziladi.

a) birlashmasi, birlashmasi $A \cup B$ b) kesishmasi, kesishmasi $A \cap B$
c) ayirmasi, ayirmasi $A \backslash B$ d) kesishmasi, kesishmasi $A \cup B$

13. A va B to`plamlarning … deb bu to`plamlarning ikkalasiga ham tegishli elementlardan tuzilgan to`plamga aytiladi. $A$ va $B$ to`plamlarning … kabi yoziladi.

a) birlashmasi, birlashmasi $A \cup B$ b) kesishmasi, kesishmasi $A \cap B$
c) ayirmasi, ayirmasi $A \backslash B$ d) kesishmasi, kesishmasi $A \cup B$

14. A to`plamdan $B$ to`plamning … deb, A ga tegishli, lekin B ga tegishli bo`lmagan elementlardan iborat to`plamga aytiladi va bu … kabi yoziladi.

a) birlashmasi, birlashmasi $A \cup B$ b) kesishmasi, kesishmasi $A \cap B$
c) ayirmasi, ayirma $A \backslash B$ d) kesishmasi, kesishmasi $A \cup B$

15. Natural sonlar to`plamini ko`rsating.

a) $\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ b) $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
c) $\mathbb{Q} = \{\frac{p}{q}: p \in \mathbb{Z},\ q \in \mathbb{N}\}$ d) $\mathbb{R} = (-\infty;\ +\infty)$

16. Butun sonlar to`plamini ko`rsating.

a) $\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ b) $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
c) $\mathbb{Q} = \{\frac{p}{q}: p \in \mathbb{Z},\ q \in \mathbb{N}\}$ d) $\mathbb{R} = (-\infty;\ +\infty)$

16. Ratsional sonlar to`plamini ko`rsating.

a) $\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ b) $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
c) $\mathbb{Q} = \{\frac{p}{q}: p \in \mathbb{Z},\ q \in \mathbb{N}\}$ d) $\mathbb{R} = (-\infty;\ +\infty)$

17. Haqiqiy sonlar to`plamini ko`rsating.

a) $\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ b) $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
c) $\mathbb{Q} = \{\frac{p}{q}: p \in \mathbb{Z},\ q \in \mathbb{N}\}$ d) $\mathbb{R} = (-\infty;\ +\infty)$

18. Kompleks sonlar to`plamini ko`rsating.

a) $\mathbb{C} = \{z = x + iy: x \in \mathbb{R},\ y \in \mathbb{R}\}$ b) $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
c) $\mathbb{Q} = \{\frac{p}{q}: p \in \mathbb{Z},\ q \in \mathbb{N}\}$ d) $\mathbb{R} = (-\infty;\ +\infty)$

19. $z = x + iy$ kompleks sonda i nima deyiladi?

a) mavhum birlik b) mavhum qismi
c) haqiqiy qismi d) o`zaro qo`shma son

20. $z = x + iy$ kompleks sonda x nima deyiladi?

a) mavhum birlik b) mavhum qismi
c) haqiqiy qismi d) o`zaro qo`shma son

21. $z = x + iy$ kompleks sonda y nima deyiladi?

a) mavhum birlik b) mavhum qismi
c) haqiqiy qismi d) o`zaro qo`shma son

22. $z = x + iy$ va $\bar{z} = x - iy$ sonlar ... deyiladi.

a) mavhum birlik b) mavhum qismi
c) haqiqiy qismi d) o`zaro qo`shma sonlar

23. X va Y – sonli to`plamlarda har bir $x \in X$ elementga yagona $y \in Y$ elementni mos qo`yuvchi qonuniyat … deyiladi.

a) funksiya b) erkli o`zgaruvchi
c) aniqlanish sohasi d) qiymatlar to`plami

24. x argument qabul qilishi mumkin bo`lgan to`plam $y = f(x)$ funksiyaning … deb ataladi.

a) funksiya b) erkli o`zgaruvchi
c) aniqlanish sohasi d) qiymatlar to`plami

25. y funksiya erishishi mumkin bo`lgan sonlar to`plami $y = f(x)$ funksiyaning … deb ataladi.

a) funksiya b) erkli o`zgaruvchi
c) aniqlanish sohasi d) qiymatlar to`plami

26. $x \in D(f)$ bo`lganda $Oxy$ tekislikdagi $(x,\ f(x))$ nuqtalar to`plami $y = f(x)$ funksiyaning … deyiladi.

a) grafigi b) erkli o`zgaruvchi
c) aniqlanish sohasi d) qiymatlar to`plami

27. Agar funksiya bitta yoki bir nechta formulalar yoki tenglamalar bilan berilgan bo`lsa, u holda bu funksiya … berilgan deyiladi.

a) analitik usulda b) jadval usulida
c) grafik usulda d) murakkab funksiya

28. Agar X to`plamda $y = f(x)$ funksiya berilgan bo`lib, x argument ham T to`plamda aniqlangan $x = \varphi(t)$ funksiya bo`lsa, u holda T to`plamda $y = f\big(\varphi(t)\big)$ ... aniqlangan deyiladi.

a) teskari funksiya b) jadval usulida
c) grafik usulda d) murakkab funksiya

29. Agarda $y = f(x)$ tenglama x ga nisbatan bir qiymatli yechilsa, ya'ni shunday $x = g(y)$ funksiya topilib, $y = f\big(g(y)\big)$ bajarilsa, u holda $x = g(y)$, yoki odatdagi belgilashlarga ko`ra $y = g(x)$ funksiya berilgan $y = f(x)$ funksiyaga ... deyiladi.

a) teskari funksiya b) jadval usulida
c) grafik usulda d) murakkab funksiya

30. $y = f(x)$ funksiya y ga nisbatan yechilmagan $F(x,\ y) = 0$ tenlgama bilan berilgan bo`lsa, u holda bu funksiya … berilgan deyiladi.

a) oshkormas holda b) oshkor holda
c) teskari funksiya d) murakkab funksiya

9-§. LIMITLAR

9.1. Sonli ketma-ketlik va uning limiti (nazariy qism)

Argumenti natural sonlar bo`lgan $f: \mathbb{N} \to \mathbb{R}$ funksiya *sonli ketma-ketlik* deyiladi. $n \in \mathbb{N}$ uchun $f(n)$ qiymatni x_n orqali belgilashadi. $f: \mathbb{N} \to \mathbb{R}$ ketma-ketlikni odatda uning qiymatlari orqali $x_1, x_2, \dots, x_n, \dots$, yoki (x_n), yoki $\{x_n\}$ kabi belgilashadi. x_1 qiymat $\{x_n\}$ ketma-ketlikning *1-hadi*, x_2 – *2-had*, va hakozo x_n – *umumiy hadi* yoki *n-hadi* deyiladi.

$\{x_n\}$ ketma-ketlik va a soni berilgan bo`lsin. Agarda har qanday $\varepsilon > 0$ sonni olganda ham, $\varepsilon$ ga bog`liq shunday n_0 natural son topilib, $n > n_0$ bo`ladigan barcha n natural sonlar uchun $|x_n - a| < \varepsilon$ tengsizlik bajarilaversa, u holda a soni $\{x_n\}$ *ketma-ketlikning limiti* deyiladi va $\lim\limits_{n\to\infty} x_n$ orqali belgilashadi. Demak, bu holda $\lim\limits_{n\to\infty} x_n = a$. Ayrim hollarda $x_n \xrightarrow[n\to\infty]{} a$, yoki $x_n \to a$, $n \to \infty$ yozuvlar ham ishlatiladi.

$\lim\limits_{n\to\infty} x_n = a$ bo`lsa, $\{x_n\}$ ketma-ketlik a soniga *yaqinlashadi* deyishadi. Limitga ega ketma-ketlik *yaqinlashuvchi ketma-ketlik* deyiladi.

$n_1, n_2, \dots, n_k, \dots$ – natural sonlarning ixtiyoriy cheksiz o`suvchi ketma-ketligi bo`lsin, $\{n_1, n_2, \dots, n_k, \dots\} \subset \mathbb{N}$. Har qanday $\{x_1, x_2, \dots, x_n, \dots\}$ ketma-ketlik uchun $\{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$ to`plam uning *qism ketma-ketligi* deyiladi.

Yaqinlashuvchi ketma-ketliklar quyidagi xossalarga ega ($\{x_n\}$ va $\{y_n\}$ yaqinlashuvchi ketma-ketliklar, $k \in \mathbb{R}$) :

1°. Yaqinlashuvchi ketma-ketlik yagona limitga ega.

2°. $\lim\limits_{n\to\infty}(x_n \pm y_n) = \lim\limits_{n\to\infty} x_n \pm \lim\limits_{n\to\infty} y_n$.

3°. $\lim\limits_{n\to\infty} kx_n = k \lim\limits_{n\to\infty} x_n$.

4°. $\lim\limits_{n\to\infty}(x_n \cdot y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.

5°. $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$, $\qquad \lim\limits_{n\to\infty} y_n \neq 0$.

6°. Agar biror n dan boshlab, biror b son uchun $x_n \geq b$ (yoki $x_n \leq b$) bo`lsa, u holda $\lim\limits_{n\to\infty} x_n \geq b$ (yoki, mos ravishda $\lim\limits_{n\to\infty} x_n \leq b$) bo`ladi.

9.2. Monoton ketma-ketlikning limiti

$\{x_n\}$ ketma-ketlik berilgan bo`lsin. Agar barcha n sonlar uchun:

$x_n < x_{n+1}$ bo`lsa, $\{x_n\}$ *o`suvchi ketma-ketlik*;

$x_n > x_{n+1}$ bo`lsa, $\{x_n\}$ *kamayuvchi ketma-ketlik*;

$x_n \geq x_{n+1}$ bo`lsa, $\{x_n\}$ *o`smovchi ketma-ketlik*;

$x_n \leq x_{n+1}$ bo`lsa, $\{x_n\}$ *kamaymovchi ketma-ketlik*

deyiladi.

O`suvchi, kamayuvchi, o`smovchi va kamaymovchi ketma-ketliklar umumiy nom bilan *monoton ketma-ketliklar* deyiladi.

$\{x_n\}$ ketma-ketlik uchun shunday M soni topilib, ixtiyoriy n uchun $x_n \leq M$ bo`lsa, $\{x_n\}$ *yuqoridan chegaralangan ketma-ketlik* deyiladi.

$\{x_n\}$ ketma-ketlik uchun shunday M soni topilib, ixtiyoriy n uchun $x_n \geq M$ bo`lsa, $\{x_n\}$ *quyidan chegaralangan ketma-ketlik* deyiladi.

Ham quyidan, ham yuqoridan chegaralangan ketma-ketlik *chegaralangan ketma-ketlik* deyiladi. Boshqacha aytganda, $\{x_n\}$ ketma-ketlik uchun shunday M soni topilib, ixtiyoriy n uchun $|x_n| \leq M$ bo`lsa, $\{x_n\}$ *chegaralangan ketma-ketlik* deyiladi.

Har qanday yaqinlashuvchi ketma-ketlik chegaralangan bo`ladi.

Chegaralangan ketma-ketliklar ushbu ajoyib **xossa**ga ega:

Har qanday chegaralangan ketma-ketlikdan yaqinlashuvchi qism ketma-ketlik ajratish mumkin.

Agar $\{x_1,\ x_2, \dots ,\ x_n, \dots\}$ – yaqinlashuvchi ketma-ketlik bo`lsa, uning ixtiyoriy $\{x_{n_1},\ x_{n_2}, \dots ,\ x_{n_k}, \dots\}$ qism ketma-ketligi ham yaqinlashuvchi bo`ladi, vaholanki, $\lim\limits_{k\to\infty} x_{n_k} = \lim\limits_{n\to\infty} x_n$. Aksinchasi o`rinli emas, ya'ni qism ketma-ketlik yaqinlashuvchi ekanligidan ketma-ketlikning yaqinlashuvchi bo`lishi kelib chiqmaydi.

Chegaralangan ketma-ketliklar uchun quyidagi **tasdiqlar** o`rinli:

**O`suvchi** (yoki **kamaymovchi**) ketma-ketlik **yuqoridan** chegaralangan bo`lsa, bu ketme-ketlik limitga ega;

Kamayuvchi (yoki **o`smovchi**) ketma-ketlik **quyidan** chegaralangan bo`lsa, bu ketme-ketlik limitga ega.

$\{x_n\}$ ketma-ketlik uchun shunday M soni topilib, ixtiyoriy n uchun $|x_n| \geq M$ bo`lsa, $\{x_n\}$ *chegaralanmagan ketma-ketlik* deyiladi. Bu hol

$$\lim_{n\to\infty} x_n = \infty$$

ko`rinishda yoziladi.

$\{x_n\}$ chegaralanmagan ketma-ketlik bo`lib, biror n dan boshlab, bu ketma-ketlikning barcha hadlari:

musbat bo`lsa, $\lim\limits_{n\to\infty} x_n = +\infty$ yozuv;

manfiy bo`lsa, $\lim\limits_{n\to\infty} x_n = -\infty$ yozuv

ishlatiladi.

$\{x_n\}$ ketma-ketlik limitga ega bo`lmasa yoki $\lim\limits_{n\to\infty} x_n = \infty$ bo`lsa, bu ketma-ketlik *uzoqlashuvchi ketma-ketlik* deyiladi.

Yaqinlashuvchi ketma-ketliklar uchun bayon etilgan tasdiqni ushbu

$$x_n = \left(1 + \frac{1}{n}\right)^n, \qquad n \in \mathbb{N}\,.$$

ketma-ketlik uchun qo`llaymiz. Bu ketma-ketlik o`suvchi, ya'ni barcha $n \in \mathbb{N}$ lar uchun $x_n < x_{n+1}$ bo`ladi. Undan tashqari, bu ketma-ketlik yuqoridan chegaralangan, masalan, 3 sonini olsak, barcha $n \in \mathbb{N}$ lar uchun $x_n < 3$ bo`ladi. Demak, $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ ketma-ketlik yaqinlashuvchi, ya'ni $\lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ limit mavjud bo`ladi. Bu limit e orqali belgilanadi va Neper soni deyiladi.Shunday qilib,

$\lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \mathrm{e}.$

9.3. Funksiyaning nuqtadagi va cheksizlikdagi limitlari

$y = f(x)$ funksiya, uning aniqlanish sohasidan olingan a nuqta berilgan bo`lsin.

Funksiyaning nuqtadagi limitining quyidagi ta'rifi Koshi tomonidan kiritilgan.

Agarda har qanday $\varepsilon > 0$ sonni olganda ham, shunday $\delta > 0$ son topilib, $|x - a| < \delta$ tengsizlikni qanoatlantiruvchi, a dan farqli barcha x lar uchun $|f(x) - b| < \varepsilon$ tengsizlik bajarilaversa, u holda b son $y = f(x)$ funksiyaning $x = a$ *nuqtadagi limiti* deyiladi va $\lim\limits_{x \to a} f(x) = b$ kabi belgilanadi.

Quyidagi ta'rif Geyne tomonidan kiritilgan.

$y = f(x)$ funksiya, uning aniqlanish sohasidan olingan a nuqta berilgan bo`lsin.

Barcha hadlari a sonidan farqli va a ga intiluvchi ixtiyoriy $\{x_n\} \subset D(f)$ ketma-ketlikni olganda ham $\{f(x_n)\}$ ketma-ketlik b soniga intilaversa, u holda b soni $y = f(x)$ funksiyaning $x = a$ nuqtadagi limiti deyiladi.

Uzluksizlikning Koshi va Geyne ta'riflari teng kuchli.

Agarda har qanday $\varepsilon > 0$ sonni olganda ham, shunday musbat N son topilib, $|x| > N$ tengsizlikni qanoatlantiruvchi barcha x lar uchun $|f(x) - b| < \varepsilon$ tengsizlik bajarilaversa, u holda b son $y = f(x)$ funksiyaning *cheksizlikdagi limiti* deyiladi va $\lim\limits_{x \to \infty} f(x) = b$ kabi belgilanadi. Agarda barcha x lar:

$x > N$ bo`lsa, $\lim\limits_{x \to +\infty} f(x) = b$ yozuv,

$x < -N$ bo`lsa, $\lim\limits_{x \to -\infty} f(x) = b$ yozuv

ishlatiladi.

9.4. Bir tomonli limitlar

x o`zgaruvchining $a$ kichik bo`lib a ga yaqinlashadigan holati $x \to a - 0$ kabi belgilanadi va x o`zgaruvchi a ga *chapdan yaqinlashadi* deyiladi.

x o`zgaruvchining $a$ katta bo`lib a ga yaqinlashadigan holati $x \to a + 0$ kabi belgilanadi va x o`zgaruvchi $a$ ga *o`ngdan yaqinlashadi* deyiladi.

$a = 0$ bo`lsa, mos ravishda $x \to 0 -$ yoki $x \to 0 +$ belgilashlar ishlatiladi.

Agarda:

$\lim\limits_{x \to a-0} f(x)$ mavjud bol`sa, u holda uni $y = f(x)$ funksiyaning $x = a$ *nuqtadagi chap tomonli limiti*;

$\lim\limits_{x \to a+0} f(x)$ mavjud bol`sa, u holda uni $y = f(x)$ funksiyaning $x = a$ *nuqtadagi o`ng tomonli limiti* deyiladi

Chap tomonli va o`ng tomonli limitlar *bir tomonli limitlar* deyiladi.

9.5. Cheksiz kichik va cheksiz katta miqdorlar

Agar $\lim\limits_{x\to a}\alpha(x)=0$ bo`lsa, u holda $\alpha(x)$ funksiya $a$ nuqtada *cheksiz kichik funksiya* deyiladi. Odatda cheksiz kichik funksiya iborasi o`rniga *cheksiz kichik miqdor* iborasi ishlatiladi.

a nuqtada cheksiz kichik miqdorlar quyidagi xossalarga ega:

1°. $\lim\limits_{x\to a}\alpha(x)=0$ va $\lim\limits_{x\to a}\beta(x)=0$ bo`lsa, $\lim\limits_{x\to a}\big(\alpha(x)\pm\beta(x)\big)=0$ bo`ladi.

2°. $\lim\limits_{x\to a}\alpha(x)=0$ va $\lim\limits_{x\to a}\beta(x)=0$ bo`lsa, $\lim\limits_{x\to a}\big(\alpha(x)\cdot\beta(x)\big)=0$ bo`ladi.

3°. $\lim\limits_{x\to a}\alpha(x)=0$ va $f(x)$ chegaralangan bo`lsa, $\lim\limits_{x\to a}\big(\alpha(x)\cdot f(x)\big)=0$ bo`ladi.

$3^{\circ\circ}$. Xususan, $\lim\limits_{x\to a}\alpha(x)=0$ bo`lsa, $\lim\limits_{x\to a}\big(C\cdot\alpha(x)\big)=0$, bu yerda C –son.

$y=f(x)$ funksiya, uning aniqlanish sohasidan olingan a nuqta berilgan bo`lsin.

a ga intiluvchi ixtiyoriy $\{x_n\}\subset D(f)$ ketma-ketlikni va har qanday musbat N sonni olganda ham shunday $n_0=n(N)$ nomer topilib, barcha $n>n_0$ sonlar uchun $|f(x_n)|>N$ tengsizlik bajarilaversa, u holda $f(x)$ funksiya a nuqtada *cheksiz katta funksiya* (yoki *cheksiz katta miqdor*) deyiladi. Bu holat quyidagicha yoziladi:
$\lim\limits_{x\to a}f(x)=\infty$.

Cheksiz katta miqdorlar cheksiz kichik miqdorlarning 1° xossadan tashqari barcha xossalariga ega.

Ravshanki, $f(x)$ funksiya a nuqtada cheksiz katta miqdor bo`lsa, u holda $\frac{1}{f(x)}$ funksiya $a$ nuqtada cheksiz kichik miqdor bo`ladi.

$\alpha(x)$ va $\beta(x)$ funksiyalar a nuqtada cheksiz kichik miqdorlar bo`lsin.

1) Agar $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)}=0$ bo`lsa, $\alpha(x)$ miqdor $a$ nuqtada $\beta(x)$ miqdorga *qaraganda yuqori tartibli* cheksiz kichik miqdor deyiladi. Bu holat $\alpha(x)=o(\beta(x))$ kabi yoziladi va “ $\alpha(x)$ miqdor $\beta(x)$ dan o-kichik” deb o`qiladi;

2) $A\neq 0$ bo`lib, $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)}=A$ bo`lsa, $\alpha(x)$ va $\beta(x)$ lar a nuqtada *bir xil tartibli* cheksiz kichik miqdorlar deyiladi.

Agar $x=a$ nuqtaning atrofida $\left|\frac{\alpha(x)}{\beta(x)}\right|$ nisbat chegaralangan bo`lsa, $\alpha(x)$ miqdor $a$ nuqtada *xuddi* $\beta(x)$ miqdor *kabi kamayadi* deyiladi va $\alpha(x)=O(\beta(x))$ kabi yoziladi. $\alpha(x)=O(\beta(x))$ tenglik “$\alpha(x)$ miqdor $\beta(x)$ dan O-katta” kabi o`qiladi;

3) Agar a nuqtada $\frac{\alpha(x)}{\beta(x)}$ nisbatning limiti mavjud bo`lmasa, $\alpha(x)$ va $\beta(x)$ lar a nuqtada *taqqoslanmaydigan* cheksiz kichik miqdorlar deyiladi;

4) Agar $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)}=1$ bo`lsa, u holda $\alpha(x)$ va $\beta(x)$ miqdorlar a nuqtada *ekvivalent* cheksiz kichik miqdorlar deyiladi hamda $\alpha(x)\sim\beta(x)$ belgilash ishlatiladi.

Quyidagi funksiyalar jufti $x\to 0$ da ekvivalent cheksiz kichik miqdorlar bo`ladi:

1. $\sin x\sim x$,
2. $\operatorname{tg} x\sim x$,
3. $\arcsin x\sim x$,
4. $\operatorname{arctg} x\sim x$,
5. $\ln(x+1)\sim x$,
6. $\log_a(x+1)\sim\frac{x}{\ln a}$,

7. $(e^x - 1) \sim x$, 8. $(a^x - 1) \sim x \ln a$,

9. $(1 - \cos x) \sim \frac{x^2}{2}$, 10. $(1 + x)^\alpha \sim \alpha x$.

Ekvivalent cheksiz kichik miqdorlarning ajoyib xossalaridan biri ushbudan iborat:

$\alpha(x)$, $\beta(x)$, $\alpha_1(x)$, $\beta_1(x)$ lar $x \to a$ da cheksiz kichik miqdorlar hamda $\alpha(x) \sim \alpha_1(x)$ va $\beta(x) \sim \beta_1(x)$ bo`lsin. U holda $\lim\limits_{x \to a} \frac{\alpha(x)}{\beta(x)} = \lim\limits_{x \to a} \frac{\alpha_1(x)}{\beta_1(x)}$.

9.6. Limitlar nazariyasining asosiy xossalari

Funksiyaning nuqtadagi limitining asosiy xossalarini keltiramiz.

1°. $f(x)$ funksiya $x \to a$ da limitga ega bo`lsa, bu limit yagona bo`ladi.

2°. O`zgarmas $C$ ning har qanday $a$ nuqtadagi limiti o`ziga teng: $\lim\limits_{x \to a} C = C$.

3°. $f(x)$ funksiya $x \to a$ da limitga ega bo`lsa,
$\lim\limits_{x \to a} kf(x) = k \lim\limits_{x \to a} f(x)$ bo`ladi, bu yerda k – ixtiyoriy son.

4°. $f(x)$ va $g(x)$ funksiyalar $x \to a$ da limitga ega bo`lsa,
$\lim\limits_{x \to a} (f(x) \pm g(x)) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$ bo`ladi.

5°. $f(x)$ va $g(x)$ funksiyalar $x \to a$ da limitga ega bo`lsa,
$\lim\limits_{x \to a} (f(x) \cdot g(x)) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$ bo`ladi.

6°. $f(x)$ va $g(x)$ funksiyalar $x \to a$ da limitga ega hamda $\lim\limits_{x \to a} g(x) \neq 0$ bo`lsa,
$\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ bo`ladi.

7°. Agar $f(x)$ funksiya $x \to a$ da limitga ega va $f(x) \geq 0$ (yoki $f(x) \leq 0$) bo`lsa, $\lim\limits_{x \to a} f(x) \geq 0$ (mos ravishda $\lim\limits_{x \to a} f(x) \leq 0$) bo`ladi.

8°. $f(x)$, $g(x)$, $h(x)$ funksiyalar uchun $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = A$ bo`lib, $a$ nuqtaning biror atrofida $f(x) \leq g(x) \leq h(x)$ bo`lsa, $\lim\limits_{x \to a} g(x) = A$ bo`ladi.

9°. $f(x)$ va $g(x)$ funksiyalar $x \to a$ da limitga ega bo`lsa,
$\lim\limits_{x \to a} \left(f(x)\right)^{g(x)} = \left(\lim\limits_{x \to a} f(x)\right)^{\lim\limits_{x \to a} g(x)}$ bo`ladi.

Funksiyaning nuqtadagi limitining yuqoridagi xossalari funksiyaning cheksizlikdagi limitlari uchun ham o`rinli.

9.7. Aniqmasliklar

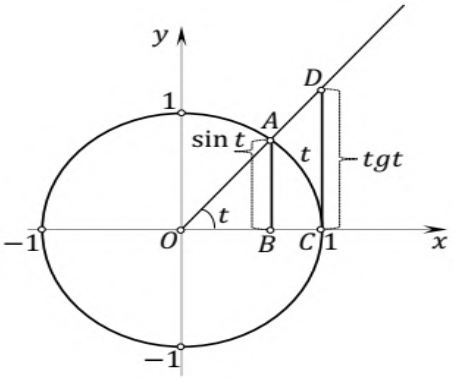
1. $\lim\limits_{x \to a} f(x) = 0$ va $\lim\limits_{x \to a} g(x) = 0$ bo`lsin. U holda bu limitlarning $\frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ nisbati $\mathbf{\frac{0}{0}}$ tipidagi aniqmaslik deyiladi.

2. $\lim\limits_{x\to a} f(x) = \infty$ va $\lim\limits_{x\to a} g(x) = \infty$ bo`lsin. U holda bu limitlarning $\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ nisbati $\frac{\infty}{\infty}$ tipidagi aniqmaslik deyiladi.

3. $\lim\limits_{x\to a} f(x) = 0$ va $\lim\limits_{x\to a} g(x) = 0$ bo`lsin. U holda bu limitlarning $\left(\lim\limits_{x\to a} f(x)\right)^{\lim\limits_{x\to a} g(x)}$ darajasi $\mathbf{0^0}$ tipidagi aniqmaslik deyiladi.

4. $\lim\limits_{x\to a} f(x) = \infty$ va $\lim\limits_{x\to a} g(x) = \infty$ bo`lsin. U holda bu limitlarning $\lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$ ayirmasi $\infty - \infty$ tipidagi aniqmaslik deyiladi.

5. $\lim\limits_{x\to a} f(x) = \infty$ va $\lim\limits_{x\to a} g(x) = 0$ bo`lsin. U holda bu limitlarning $\lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$ ko`paytmasi $\infty \cdot \mathbf{0}$ tipidagi aniqmaslik deyiladi.



6. $\lim\limits_{x\to a} f(x) = 1$ va $\lim\limits_{x\to a} g(x) = \infty$ bo`lsin. U holda bu limitlarning $\left(\lim\limits_{x\to a} f(x)\right)^{\lim\limits_{x\to a} g(x)}$ darajasi $\mathbf{1}^\infty$ tipidagi aniqmaslik deyiladi.

7. $\lim\limits_{x\to a} f(x) = \infty$ va $\lim\limits_{x\to a} g(x) = 0$ bo`lsin. U holda bu limitlarning $\left(\lim\limits_{x\to a} f(x)\right)^{\lim\limits_{x\to a} g(x)}$ darajasi $\infty^{\mathbf{0}}$ tipidagi aniqmaslik deyiladi.

9.8. Birinchi va ikkinchi ajoyib limitlar

$t \to 0$ da $\sin t \to 0$ bo`ladi. Demak, $\frac{\sin t}{t}$ nisbatning $t \to 0$ dagi limiti $\frac{0}{0}$ tipdagi aniqmaslik ekan. Bu aniqmaslikning qiymati 1 ga teng ekanligi ko`rsatuvchi

$$\lim_{t\to 0} \frac{\sin t}{t} = 1$$

tenglik *birinchi ajoyib limit* deb yuritiladi. Uni isbotlash uchun ushbu chizmadan foydalanamiz. Unga ko`ra, $\sin t < t < tg\, t$, yoki $\cos t < \frac{\sin t}{t} < 1$ qo`shtengsizlikka egamiz. $\lim\limits_{x\to 0} \cos x = 1$ ekanligidan talab etiladigan tenglik kelib chiqadi.

e soni $\left(1 + \frac{1}{n}\right)^n$ ketma-etlikning $n \to \infty$ dagi limiti orqali aniqlanishini ta'kidlagan edik, ya'ni $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}$.

$x \to \infty$ da $f(x) = \left(1 + \frac{1}{x}\right)^x$ funksiya limitining qiymati haqidagi savol paydo bo`ladi. Ravshanki, bu limit 1^∞ tipdagi aniqmaslikni beradi. Bu aniqmaslikning cheksizlikdagi limiti

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \mathrm{e}$$

bo`lar ekan. Bu tenglik *ikkinchi ajoyib limit* deyiladi.

Mustahkamlash uchun mashqlar

Limitlarga oid masalalar

1

1. $\{x_n\} = \left\{\frac{3n+5}{n-1}\right\}$ sonli ketma-ketlik $A = 3$ limitga ega ekanligini isbotlang.

2

Quyidagi sonli ketma-ketliklarning limitini toping.

1. $\lim\limits_{n\to\infty} \frac{3n^2+3n-5}{1-n^2}$ 2. $\lim\limits_{x\to\mp\infty} \frac{2+4x^2+3x^2}{x^3-7x-10}$ 3. $\lim\limits_{x\to\mp\infty} \frac{7x^2+10x+20}{x^3-10x^2-1}$

4. $\lim\limits_{x\to 2} \frac{x^3-3x^2+3}{x^2-3}$ 5. $\lim\limits_{x\to 2} \frac{x^2-7x+10}{8-x^3}$ 6. $\lim\limits_{x\to 1} \frac{x^3-3x^2+2}{x^2-7x+6}$

7. $\lim\limits_{x\to 2} \frac{\sqrt{x+7}-3}{\sqrt{x+2}-2}$ 8. $\lim\limits_{x\to\infty} (x(\sqrt{x^2+4}-x))$ 9. $\lim\limits_{x\to 1} (\frac{1}{1-x} - \frac{3}{1-x^3})$

3

Ko`rsatilgan funksiyaning limitini toping.

1. $\lim\limits_{x\to 0} \frac{tg\, 3x}{\sin 2x}$ 2. $\lim\limits_{x\to 0} \frac{1-\cos 6x}{x \sin 3x}$

3. $\lim\limits_{x\to 1} \frac{\sin(2(x-1))}{x^2-7x+6}$ 4. $\lim\limits_{x\to\mp\infty} (\frac{x+3}{2x-1})^x$

5. $\lim\limits_{x\to\infty} \left(\frac{3x+2}{3x-1}\right)^{4x-1}$ 6. $\lim\limits_{x\to\infty} ((2x+1)(\ln(3x+1) - \ln(3x-2)))$

7. $\lim\limits_{x\to 2} \frac{\sin 3(x-2)}{x^2-3x+2}$ 8. $\lim\limits_{x\to 0} \frac{e^{5x}-1}{\sin 10x}$

9. $\lim\limits_{x\to 0} \frac{\ln(1+7x)}{\sin 7x}$ 10. $\lim\limits_{x\to 0} \frac{tg\, 8x}{\sin 3x}$

Ko`rsatilgan funksiyaning limitini toping.

1. $\lim\limits_{x\to 2} \frac{x^2-5x+6}{x^2-12x+20}$ 2. $\lim\limits_{x\to 0} \frac{x^3-x^2+2x}{x^2+x}$ 3. $\lim\limits_{x\to 3} \frac{6+x-x^2}{x^3-27}$

4. $\lim\limits_{x\to 1} \frac{2x^2-x-1}{3x^2-x-2}$ 5. $\lim\limits_{x\to 2} \frac{2x^2-7x+4}{x^2-5x+6}$ 6. $\lim\limits_{x\to 3} \frac{12+x-x^2}{x^3-27}$

7. $\lim\limits_{x\to 1/3} \frac{3x^2+2x-1}{27x^3-1}$ 8. $\lim\limits_{x\to -1} \frac{x^2-4x-5}{x^2-2x-3}$ 9. $\lim\limits_{x\to -1} \frac{3x^2+2x-1}{-x^2+x+2}$

10. $\lim\limits_{x\to 3} \frac{3x^2-11x+6}{2x^2-5x-3}$ 11. $\lim\limits_{x\to 2} \frac{x^3-8}{x^2+x-6}$ 12. $\lim\limits_{x\to -1} \frac{x^2x-2}{x^3+1}$

13. $\lim\limits_{x\to -3} \frac{2x^2+11x+15}{3x^2+5x-12}$ 14. $\lim\limits_{x\to 1} \frac{2x^2+5x-10}{x^3-1}$ 15. $\lim\limits_{x\to 1} \frac{x^3-3x+2}{x^2-4x+3}$

16. $\lim\limits_{x\to 2} \frac{3x^2+2x+1}{x^3-8}$ 17. $\lim\limits_{x\to 1} \frac{x^4-x^2+x+1}{x^4+1}$ 18. $\lim\limits_{x\to 1} \frac{2x^2-3x-1}{x^4-1}$

19. $\lim\limits_{x\to 2} \frac{x^2-x+3}{5x^2+3x-3}$ 20. $\lim\limits_{x\to -2} \frac{x^2 2x}{x^2+4x+4}$ 21. $\lim\limits_{x\to -1} \frac{x^2-1}{x^2+3x+2}$

22. $\lim\limits_{x\to\infty} \frac{3x^3-5x^2+2}{2x^3+5x^2-x}$ 23. $\lim\limits_{x\to\infty} \frac{4x^3+7x}{2x^3-4x^2+5}$ 24. $\lim\limits_{x\to\infty} \frac{5x^4-3x^2+7}{x^4+2x^3+1}$

25. $\lim\limits_{x\to\infty} \frac{7x^3-2x^2+4x}{2x^3+5}$ 26. $\lim\limits_{x\to\infty} \frac{x^3-4x^2+28x}{5x^3+3x^2+x-1}$ 27. $\lim\limits_{x\to\infty} \frac{3x^2+10x+3}{2x^2+5x-3}$

28. $\lim\limits_{x\to\infty} \frac{-3x^4+x^2+x}{x^4+3x-2}$ 29. $\lim\limits_{x\to\infty} \frac{2x^2+7x+3}{5x^2-3x+4}$ 30. $\lim\limits_{x\to\infty} \frac{-x^2+3x+1}{3x^2+x-5}$

4

Cheksiz kichik miqdorlardan foydalanib funksiyaning limitini toping.

1. $\lim\limits_{n\to 0}\frac{ln(1+3x^2)}{x^3-5x^2}$ 2. $\lim\limits_{n\to 0}\frac{arcsin5x}{tg3x}$ 3. $\lim\limits_{n\to 0}\frac{sin7x}{tg2x}$.

4. $\lim\limits_{n\to 0}\frac{e^{3x}-1}{x^3+27x}$ 5. $\lim\limits_{n\to 0}\frac{arctg6x}{2x^2-3x}$ 6. $\lim\limits_{n\to 0}\frac{arcsin3x}{2x}$.

7. $\lim\limits_{n\to 0}\frac{sin5x}{arctg2x}$. 8. $\lim\limits_{n\to 0}\frac{ln(1+3x)}{sin2x}$. 9. $\lim\limits_{n\to 0}\frac{e^{2x}-1}{tg3x}$.

10. $\lim\limits_{n\to 0}\frac{sin\,(x-3)}{x^2-5x+6}$. 11. $\lim\limits_{n\to 0}\frac{cos3x-cosx}{2x^2}$. 12. $\lim\limits_{n\to 0}\frac{1-cos6x}{4x^2}$.

O`z-o`zini tekshirish uchun savollar

1. Sonli ketma-ketlik va uning limitini tushuntirib bering.
2. Monoton ketma-ketlikning limitini tushuntirib bering.
3. Funksiyaning nuqtadagi va cheksizlikdagi limitlarini ayting.
4. Bir tomonli limitlarni tushuntirib bering.
5. Cheksiz kichik va cheksiz katta miqdorlar deb nimaga aytiladi?
6. Limitlar nazariyasining asosiy xossalarini ayting.
7. Qanday aniqmasliklar mavjud?
8. Birinchi va ikkinchi ajoyib limitlarni yozing.

O`z-o`zini tekshirish uchun testlar

Limitlarga oid testlar

1. Argumenti natural sonlar bo`lgan f: $\mathbb{N}\to\mathbb{R}$ funksiya ... deyiladi.
a) sonli ketma-ketlik b) ketma-ketlikning limiti
c) yaqinlashuvchi ketma-ketlik d) natural sonlar

2. Agarda har qanday $\varepsilon>0$ sonni olganda ham, ε ga bog`liq shunday $n_0$ natural son topilib, $n>n_0$ bo`ladigan barcha n natural sonlar uchun $|x_n-a|<\varepsilon$ tengsizlik bajarilaversa, u holda a soni $\{x_n\}$ ... deyiladi.
a) sonli ketma-ketlik b) ketma-ketlikning limiti
c) yaqinlashuvchi ketma-ketlik d) natural sonlar

3. $\lim\limits_{n\to\infty}x_n=a$ bo`lsa, $\{x_n\}$ ketma-ketlik a soniga ... deyiladi.
a) yaqinlashadi b) uzoqlashadi c) ketma-ketlik d) natural son

4. Limitga ega ketma-ketlik ... deyiladi.
a) uzoqlashuvchi b) cheksiz
c) yaqinlashuvchi ketma-ketlik d) natural sonlar

5. Yaqinlashuvchi ketma-ketlik ...
a) yagona limitga ega. b) kamida ikkita limitga ega.
c) limitga ega emas. d) uchta limitga ega.

6. $\{x_n\}$ va $\{y_n\}$ yaqinlashuvchi ketma-ketliklar ... tenglik o`rinli.

a) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \lim\limits_{n\to\infty} x_n \pm \lim\limits_{n\to\infty} y_n$.
b) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \mathrm{k}(\lim\limits_{n\to\infty} x_n \pm \lim\limits_{n\to\infty} y_n)$.
c) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.
d) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \mathrm{k} \cdot \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.

7. $\{x_n\}$ va $\{y_n\}$ yaqinlashuvchi ketma-ketliklar ... to`g`ri tenglik toping.
a) $\lim\limits_{n\to\infty} kx_n = k \lim\limits_{n\to\infty} x_n$.
b) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \mathrm{k}(\lim\limits_{n\to\infty} x_n \pm \lim\limits_{n\to\infty} y_n)$.
c) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.
d) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \mathrm{k} \cdot \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.

8. $\{x_n\}$ va $\{y_n\}$ yaqinlashuvchi ketma-ketliklar ... to`g`ri tenglik toping.
a) $\lim\limits_{n\to\infty}(x_n \cdot y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.
b) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \mathrm{k}(\lim\limits_{n\to\infty} x_n \pm \lim\limits_{n\to\infty} y_n)$.
c) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.
d) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \mathrm{k} \cdot \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.

9. $\{x_n\}$ va $\{y_n\}$ yaqinlashuvchi ketma-ketliklar ... to`g`ri tenglik toping.
a) $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$, $\qquad \lim\limits_{n\to\infty} y_n \neq 0$.
b) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \mathrm{k}(\lim\limits_{n\to\infty} x_n \pm \lim\limits_{n\to\infty} y_n)$.
c) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.
d) $\lim\limits_{n\to\infty}(x_n \pm y_n) = \mathrm{k} \cdot \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.

10. Agar$\{x_n\}$ ketma-ketlikda barcha n sonlar uchun $x_n < x_{n+1}$ bo`lsa, $\{x_n\}$ ... deyiladi.
a) o`suvchi ketma-ketlik b) kamayuvchi ketma-ketlik
c) o`smovchi ketma-ketlik d) kamaymovchi ketma-ketlik

11. Agar$\{x_n\}$ ketma-ketlikda barcha n sonlar uchun $x_n > x_{n+1}$ bo`lsa, $\{x_n\}$ ... deyiladi.
a) o`suvchi ketma-ketlik b) kamayuvchi ketma-ketlik
c) o`smovchi ketma-ketlik d) kamaymovchi ketma-ketlik

12. Agar$\{x_n\}$ ketma-ketlikda barcha n sonlar uchun $x_n \geq x_{n+1}$ bo`lsa, $\{x_n\}$ ... deyiladi.
a) o`suvchi ketma-ketlik b) kamayuvchi ketma-ketlik
c) o`smovchi ketma-ketlik d) kamaymovchi ketma-ketlik

13. Agar$\{x_n\}$ ketma-ketlikda barcha n sonlar uchun $x_n \leq x_{n+1}$ bo`lsa, $\{x_n\}$ ... deyiladi.
a) o`suvchi ketma-ketlik b) kamayuvchi ketma-ketlik
c) o`smovchi ketma-ketlik d) kamaymovchi ketma-ketlik

14. O`suvchi, kamayuvchi, o`smovchi va kamaymovchi ketma-ketliklar umumiy nom bilan ... deyiladi.

a) monoton ketma-ketliklar b) chegaralangan ketma-ketlik
c) yuqoridan chegaralangan ketma-ketlik
d) quyidan chegaralangan ketma-ketlik

15. $\{x_n\}$ ketma-ketlik uchun shunday M soni topilib, ixtiyoriy n uchun $x_n \leq M$ bo`lsa, $\{x_n\}$ ... deyiladi.
a) monoton ketma-ketliklar b) chegaralangan ketma-ketlik
c) yuqoridan chegaralangan ketma-ketlik
d) quyidan chegaralangan ketma-ketlik

16. $\{x_n\}$ ketma-ketlik uchun shunday M soni topilib, ixtiyoriy n uchun $x_n \geq M$ bo`lsa, $\{x_n\}$ ... deyiladi.
a) monoton ketma-ketliklar b) chegaralangan ketma-ketlik
c) yuqoridan chegaralangan ketma-ketlik
d) quyidan chegaralangan ketma-ketlik

17. Ham quyidan, ham yuqoridan chegaralangan ketma-ketlik ... deyiladi.
a) monoton ketma-ketliklar b) chegaralangan ketma-ketlik
c) yuqoridan chegaralangan ketma-ketlik
d) quyidan chegaralangan ketma-ketlik

18. $\{x_n\}$ ketma-ketlik uchun shunday M soni topilib, ixtiyoriy n uchun $|x_n| \leq M$ bo`lsa, $\{x_n\}$ ... deyiladi.
a) monoton ketma-ketliklar b) chegaralangan ketma-ketlik
c) yuqoridan chegaralangan ketma-ketlik
d) quyidan chegaralangan ketma-ketlik

19. Agar $\{x_1, x_2, \dots, x_n, \dots\}$ – yaqinlashuvchi ketma-ketlik bo`lsa, uning ixtiyoriy $\{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$ qism ketma-ketligi ham ... bo`ladi.
a) uzoqlashuvchi b) cheksiz c) yaqinlashuvchi d) natural sonlar

20. Ikkinchi ajoyib limitni ko`rsating.
a) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ b) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1$
c) $\lim\limits_{x\to 0}\frac{sinx}{nx} = 1$ d) $\lim\limits_{x\to 0}\frac{sinx}{x} = 0$

20. Birinchi ajoyib limitni ko`rsating.
a) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ b) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1$
c) $\lim\limits_{x\to 0}\frac{sinx}{nx} = 1$ d) $\lim\limits_{x\to 0}\frac{sinx}{x} = 0$

21. Agarda har qanday $\varepsilon > 0$ sonni olganda ham, shunday $\delta > 0$ son topilib, $|x-a| < \delta$ tengsizlikni qanoatlantiruvchi, a dan farqli barcha x lar uchun $|f(x) - b| < \varepsilon$ tengsizlik bajarilaversa, u holda b son $y = f(x)$ funksiyaning $x = a$ ... deyiladi
a) nuqtadagi limiti b) qiymati c) hosilasi d) integrali

22. Funksiyaning $x = a$ nuqtadagi limiti qanday belgilanadi?
a) $\lim\limits_{x\to a} f(x) = b$ b) $|x-a| < \delta$
c) $|f(x) - b| < \varepsilon$ d) $y = f(x)$

23. x o`zgaruvchining $a$ kichik bo`lib a ga yaqinlashadigan holati ... kabi belgilanadi va x o`zgaruvchi a ga ... deyiladi.

a) $x \to a - 0$; chapdan yaqinlashadi
b) $x \to a + 0$; chapdan yaqinlashadi
c) $x \to a + 0$; o`ngdan yaqinlashadi
d) $x \to a - 0$; o`ngdan yaqinlashadi

24. x o`zgaruvchining $a$ katta bo`lib a ga yaqinlashadigan holati ... kabi belgilanadi va x o`zgaruvchi a ga ... deyiladi.

a) $x \to a - 0$; chapdan yaqinlashadi
b) $x \to a + 0$; chapdan yaqinlashadi
c) $x \to a + 0$; o`ngdan yaqinlashadi
d) $x \to a - 0$; o`ngdan yaqinlashadi

25. Agarda $\lim\limits_{x \to a-0} f(x)$ mavjud bol`sa, u holda uni $y = f(x)$ funksiyaning $x = a$ ... deyiladi

a) nuqtadagi chap tomonli limiti b) nuqtadagi o`ng tomonli limiti
a) nuqtadagi chap tomonli qiymati a) nuqtadagi o`ng tomonli qiymati

26. Agarda $\lim\limits_{x \to a+0} f(x)$ mavjud bol`sa, u holda uni $y = f(x)$ funksiyaning $x = a$ ... deyiladi

a) nuqtadagi chap tomonli limiti b) nuqtadagi o`ng tomonli limiti
c) nuqtadagi chap tomonli qiymati d) nuqtadagi o`ng tomonli qiymati

27. Chap tomonli va o`ng tomonli limitlar ... deyiladi.

a) bir tomonli limitlar b) katta limitlar
c) katta tomonli limitlar d) uch tomonli limitlar

28. Agar $\lim\limits_{x \to a} \alpha(x) = 0$ bo`lsa, u holda $\alpha(x)$ funksiya a nuqtada ... deyiladi.

a) cheksiz kichik funksiya b) cheksiz katta funksiya
c) cheksiz uzoq funksiya d) cheksiz funksiya

29. Agar $f(x)$ va $g(x)$ funksiyalar $x \to a$ da limitga ega bo`lsa, u holda ... bo`ladi.

a) $\lim\limits_{x \to a}(f(x) \pm g(x)) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$
b) $\lim\limits_{x \to a}(f(x) \pm g(x)) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$
c) $\lim\limits_{x \to a}(f(x) \pm g(x)) = k(\lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x))$
d) $\lim\limits_{x \to a}(f(x) \pm g(x)) = k(\lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x))$

30. $\lim\limits_{x \to a} f(x) = 0$ va $\lim\limits_{x \to a} g(x) = 0$ bo`la, u holda bu limitlarning $\frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ nisbati ... tipidagi aniqmaslik deyiladi.

a) $\frac{0}{0}$ b) $\frac{\infty}{\infty}$ c) 0^0 d) $\infty - \infty$

10-§. УЗЛУКСИЗ ФУНКЦИЯЛАР

10.1. Funksiyaning nuqtadagi uzluksizligi (nazariy qism)

$f(x)$ funksiya X sohada aniqlangan va $\xi \in X$ uning biror *ichki nuqtasi*, ya'ni shunday $(a,\ b)$ interval topiladiki, $\xi \in (a,\ b) \subset X$ munosabatlarni qanoatlantiradigan nuqta bo`lsin.

Ixtiyoriy $\varepsilon > 0$ sonni olganda ham, shunday $\delta > 0$ son topilib, $|x - \xi| < \delta$ tengsizlikni qanoatlantiruvchi har bir x nuqta uchun $|f(x) - f(\xi)| < \varepsilon$ tengsizlikning bajarilishi kelib chiqaversa, u holda $y = f(x)$ funksiya ξ *nuqtada uzluksiz* deyiladi.

Uzluksizning boshqa ta'rifi. ξ nuqtaga yaqinlashuvchi ixtiyoriy $\{x_n\} \subset X$ ketma-ketlikni olganda ham, mos qiymatlardan tuzilgan $\{f(x_n)\}$ ketma-ketlik $f(\xi)$ qiymatga intilaversa, u holda $y = f(x)$ funksiya ξ *nuqtada uzluksiz* deyiladi.

Uzluksizlikni qisqacha quyidagicha ta'riflash mumkin: agar

$$\lim_{x\to\xi} f(x) = f(\xi)$$

bo`lsa, u holda $y = f(x)$ funksiya ξ *nuqtada uzluksiz* deyiladi.

Bu ta'riflar teng kuchli.

Uzluksizlikning amaliy masalalarni yechishda keng qo`llaniladigan ushbu alomatini keltiramiz:

$$\lim_{x\to\xi} f(x) = f\left(\lim_{x\to\xi} x\right).$$

Agar $\lim\limits_{x\to\xi-0} f(x) = f(\xi)$ bo`lsa, $f(x)$ funksiya $\xi$ nuqtada *o`ngdan uzluksiz* deyiladi.

Agar $\lim\limits_{x\to\xi+0} f(x) = f(\xi)$ bo`lsa, $f(x)$ funksiya ξ nuqtada *chapdan uzluksiz* deyiladi.

Uzluksiz funksiya ham chapdan, ham o`ngdan uzluksiz bo`ladi.

Aksinchasi, birgalikda ham chapdan, ham o`ngdan uzluksiz funksiya uzluksiz bo`ladi.

10.1.1.-misol. $f(x) = \begin{cases} 1 - x, & x < 0, \\ x - 1, & x \geq 0 \end{cases}$ funksiya o`ngdan uzluksiz, ya'ni

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (x - 1) = -1 = f(0),$$

lekin chapdan uzluksiz emas

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (1 - x) = 1 \neq -1 = f(0).$$

Demak, qaralayotgan bu funksiya $x = 0$ nuqtada uzluksiz emas (boshqacha aytganda, bu funksiya $x = 0$ nuqtada uzilishga ega).

Nuqtada uzluksiz funksiyalarning xossalari:

1^0. $f(x)$ va $g(x)$ funksiyalar a nuqtada uzluksiz bo`lsin. U holda

$a)$ $f(x) \pm g(x)$,

$b)$ $f(x)g(x)$,

$c)$ $\frac{f(x)}{g(x)}$

funksiyalar ham a nuqtada uzluksiz (bo`linmada $g(a) \neq 0$ shart talab etiladi).

2^0. $z = \varphi(x)$ funksiya x_0 nuqtada uxluksiz, $y = f(z)$ funksiya $z_0 = \varphi(x_0)$ nuqtada uzluksiz bo`lsa, $y = f(\varphi(x))$ murakkab funksiya $x_0$ nuqtada uzluksiz bo`ladi.

10.2. Asosiy elementar funksiyalarning uzluksizligi

Ma'lumki, *asosiy elementar funksiyalar* ustida qo`shish, ayirish, ko`paytirish, bo`lish amallarini chekli marta qo`llashdan hosil bo`lgan funksiyalar *elementar funksiyalar* deyiladi. $1^0$ xossaning $a$), $b$), $c$) punktlaridan elementar funksiyalarning uzluksizligi asosiy elementar funksiyalar uzluksizligidan kelib chiqadi. Shuning uchun quyidagi asosiy elementar funksiyalarning uzluksizligini ko`rsatish yetarli:

1. $y = x^\alpha$ darajali funksiya uzluksiz, $\alpha \in \mathbb{R}$.
2. $y = a^x$ ko`rsatkichli funksiya uzluksiz, $a \in \mathbb{R}$, $a > 0$, $a \neq 1$.
3. $y = \log_a x$ logarifmik funksiya uzluksiz, $a \in \mathbb{R}$, $a > 0$, $a \neq 1$.
4. $y = \sin x$, $y = \cos x$, $y = \operatorname{tg} x$, $y = \operatorname{ctg} x$ trigonometrik funksiyalar uzluksiz.
5. $y = \arcsin x$, $y = \arccos x$, $y = \operatorname{arctg} x$, $y = \operatorname{arcctg} x$ teskari trigonometrik funksiyalar uzluksiz.

Bu jumlalarning ayrimlarining isbotini keltiramiz.

1. $\alpha \in \mathbb{R}$ ning qiymatiga bog`liq holda $y = x^\alpha$ darajali funksiyaning aniqlanish sohasi yoki $D(\alpha) = [0, +\infty)$ yoki $D(\alpha) = (-\infty, +\infty)$ bo`ladi. $\xi \in D(\alpha)$ ixtiyoriy tayinlangan nuqta va $\{x_n\} \subset D(\alpha)$ unga yaqinlashuvchi ixtiyoriy ketma-ketlik bo`lsin, ya'ni $\lim\limits_{n\to\infty} x_n = \xi$. U holda $\lim\limits_{x\to a}\big(f(x)\big)^{g(x)} = \Big(\lim\limits_{x\to a} f(x)\Big)^{\lim\limits_{x\to a} g(x)}$ va $\lim\limits_{x\to a} C = C$ ($C$ – o`zgarmas funksiya) xossalardan $\lim\limits_{n\to\infty} x_n^\alpha = \Big(\lim\limits_{n\to\infty} x_n\Big)^\alpha = \xi^\alpha$ bo`ladi. $\{x_n\}$ ketma-ketlik $\xi$ ga yaqinlashuvchi ixtiyoriy ketma-ketlik hamda $\xi$ ning ixtiyoriy ekanligidan har bir $\alpha \in \mathbb{R}$ son uchun $y = x^\alpha$ darajali funksiya uzluksiz bo`lishi kelib chiqadi.

Topshiriq. $\lim\limits_{x\to a}\big(f(x)\big)^{g(x)} = \Big(\lim\limits_{x\to a} f(x)\Big)^{\lim\limits_{x\to a} g(x)}$ va $\lim\limits_{x\to a} C = C$ (C – o`zgarmas funksiya) xossalardan foydalanib, $y = a^x$ ko`rsatkichli funksiya uzluksiz ekanligini isbotlang, $a \in \mathbb{R}$, $a > 0$, $a \neq 1$.

3. $y = \log_a x$ logarifmik funksiya uzluksiz ekanligini isbotlaymiz. Bu yerda $a \in \mathbb{R}$, $a > 0$, $a \neq 1$. Aniqlik uchun $\mathbf{0 < a < 1}$ bo`lsin. U holda $y = \log_a x$ funksiya *kamayuvchi* bo`ladi, ya'ni x o`zgaruvchining bu funksiya aniqlanish sohasi $D = (0, +\infty)$ dan olingan ixtiyoriy x_1 va x_2 qiymatlari uchun $x_1 < x_2$ tengsizlikdan $\log_a x_1 > \log_a x_2$ tengsizlik kelib chiqadi.

Koshi ta'rifidan foydalanamiz. Bunda ixtiyoriy $\xi \in D$ nuqta uchun har qanday $\varepsilon > 0$ sonni olganda ham, shunday $\delta = \delta(\xi, \varepsilon)$ topilib, $|x - \xi| < \delta$ tengsizlikdan ushbu $|\log_a x - \log_a \xi| < \varepsilon$ tengsizlik kelib chiqishini isbotlash talab etiladi. Oxirgi tengsizlikni quyidagicha yozish mumkin

$$\left|\log_a \frac{x}{\xi}\right| < \varepsilon \Leftrightarrow -\varepsilon < \log_a \frac{x}{\xi} < \varepsilon \underset{0<a<1}{\Longleftrightarrow} a^{-\varepsilon} > \frac{x}{\xi} > a^\varepsilon \text{ yoki } \xi a^\varepsilon < x < \xi a^{-\varepsilon}.$$

Endi $\delta > 0$ ni shunday tanlaymizki, $(\xi - \delta,\ \xi + \delta) \subset (\xi a^{\varepsilon},\ \xi a^{-\varepsilon})$ bo`lsin. Buning uchun $\xi - \delta_1 = \xi a^{\varepsilon}$ va $\xi + \delta_2 = \xi a^{-\varepsilon}$ deb olib, ulardan topilgan $\delta_1 = \xi - \xi a^{\varepsilon}$ va $\delta_2 = \xi a^{-\varepsilon} - \xi$ sonlarni qaraymiz. $a \neq 1$ va $\varepsilon \neq 0$ ekanligidan $a^{\varepsilon} \neq 1$ va $a^{-\varepsilon} \neq 1$ bo`ladi. Aniqrog`i $0 < a < 1$ ekanligidan $a^{\varepsilon} < 1$ va $a^{-\varepsilon} > 1$ bo`ladi. Bundan $\delta_1 > 0$ va $\delta_2 > 0$ kelib chiqadi. Ushbuga egamiz:

$$\delta_2 = \xi a^{-\varepsilon} - \xi = \xi \cdot \left(\frac{1}{a^{\varepsilon}} - 1\right) = \xi \cdot \frac{1-a^{\varepsilon}}{a^{\varepsilon}} > \xi(1 - a^{\varepsilon}) = \delta_1,$$

ya'ni $\delta_2 > \delta_1$. Endi $\delta = \delta_1 = \xi(1 - a^{\varepsilon})$ deb olamiz. U holda

$$(\xi - \delta,\ \xi + \delta) = (\xi - \delta_1,\ \xi + \delta_1) \subset (\xi - \delta_1,\ \xi + \delta_2) \subset (\xi a^{\varepsilon},\ \xi a^{-\varepsilon}).$$

Endi Koshi ta'rifidagi shart, ya'ni ushbu implikatsiya bajarilishini tekshiramiz

$$\forall x,\ \ |\xi - x| < \delta \Rightarrow |\log_a x - \log_a \xi| < \varepsilon.$$

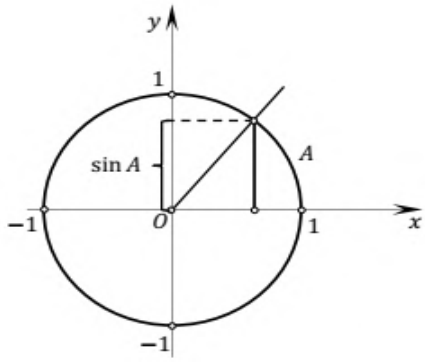
Quyidagilarga egamiz

$$\forall x,\ \ |\xi - x| < \delta \Rightarrow x \in (\xi - \delta_1, \xi + \delta_2) \Leftrightarrow \begin{cases} x > \xi - \delta_1 = \xi a^{\varepsilon}, \\ x < \xi + \delta_2 = \xi a^{-\varepsilon} \end{cases} \underset{0<a<1}{\Longleftrightarrow}$$

$$\begin{cases} \log_a x < \log_a(\xi a^{\varepsilon}) = \log_a \xi + \varepsilon, \\ \log_a x > \log_a(\xi a^{-\varepsilon}) = \log_a \xi - \varepsilon \end{cases} \Leftrightarrow \log_a \xi - \varepsilon < \log_a x < \log_a \xi + \varepsilon \Leftrightarrow$$

$$-\varepsilon < \log_a x - \log_a \xi < \varepsilon \Leftrightarrow |\log_a x - \log_a \xi| < \varepsilon.$$

Shu bilan $y = \log_a x$ logarifmik funksiyaning $0 < a < 1$ hol uzluksiz ekanligini to`la isbotlandi. Bunda $\delta = \delta(\xi,\ \varepsilon) = \xi(1 - a^{\varepsilon})$ deb olindi.



Topshiriq. $a > 1$ hol uchun $\delta = \delta(\xi,\ \varepsilon) = \xi(1 - a^{-\varepsilon})$ deb olib, $y = \log_a x$ logarifmik funksiyaning uzluksiz ekanligini isbotlang.

4. $y = \sin x$ funksiya uzluksizligini isbotlaymiz. Buning uchun

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2}$$

trigonometrik ayniyatdan, nolga yaqin A, $A \neq 0$, ifodalar uchun $|\sin A| < |A|$ hamda har qanday B uchun o`rinli bo`lgan $|\cos B| \leq 1$ tengsizliklardan foydalanamiz.

Ixtiyoriy $\xi \in \mathbb{R}$ nuqtani olamiz. Har qanday $\varepsilon > 0$ son uchun quyidagilarga egamiz

$$|x - \xi| < \varepsilon \xRightarrow[\varepsilon \text{ nolga yaqin bo`lgani uchun}]{} \left|\sin\frac{x-\xi}{2}\right| < \left|\frac{x-\xi}{2}\right| < \frac{\varepsilon}{2} \Rightarrow$$

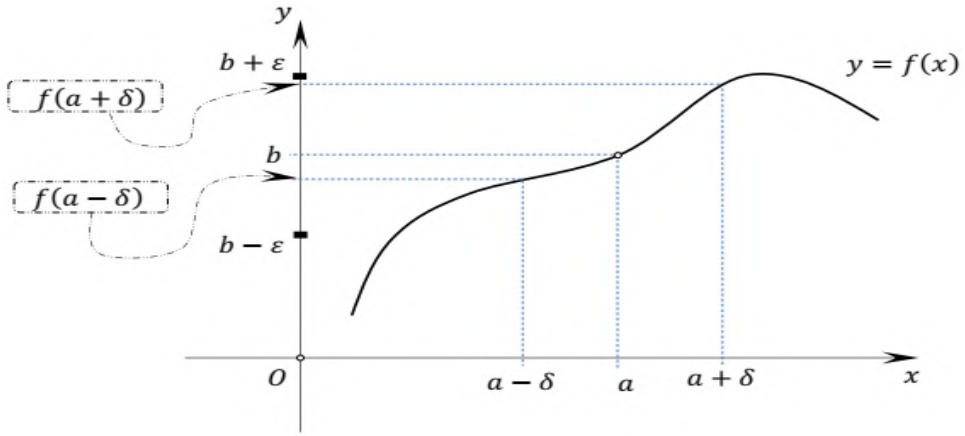
$$\Rightarrow |\sin x - \sin\xi| = \left|2\sin\frac{x-\xi}{2}\cos\frac{x+\xi}{2}\right| < 2 \cdot \frac{\varepsilon}{2} \cdot 1 = \varepsilon\ .$$

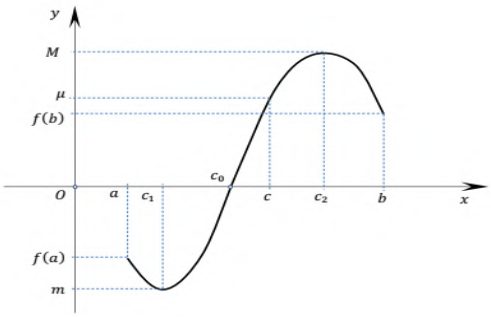
Shunday qilib, $y = \sin x$ funksiya uzluksizligini isbotladik, bunda $\delta = \delta(\xi, \varepsilon) = \varepsilon$ deb olish kifoya ekan.

Topshiriq. Boshqa trigonometrik va teskari trigonometrik funksiyalarning uzluksizligini isbotlang.

Funksiyaning nuqtadagi limiti tushunchasining geometrik talqinini keltiramiz. Quyidagi rasmda $y = f(x)$ funksiya grafigi tasvirlangan. Aytaylik, uning $x \to a$ dagi limiti b bo`lsin. Ixtiyoriy istalgancha kichik $\varepsilon > 0$ sonni olamiz va b ning ε-atrofi $(b - \varepsilon, b + \varepsilon)$ ni qaraymiz.

$b = f(a)$ bo`lsin. $Ox$ o`qida shunday $(a-\delta,\ a+\delta)$ atrofni tanlaymizki, $x \in (a-\delta,\ a+\delta)$ ekanligidan $f(x) \in (b-\varepsilon, b+\varepsilon)$ kelib chiqsin. ε kichiklashganda $(b-\varepsilon, b+\varepsilon)$ interval b ga tortiladi, unga mos $(a-\delta,\ a+\delta)$ interval esa a ga tortiladi. Bu $\lim\limits_{x\to a} f(x) = b$ holatning geometrik talqinidir.





10.3. Kesmada uzluksiz funksiyalarning asosiy xossalari

$f(x)$ funksiya $[a,\ b]$ kesmaning har bir nuqtasida uzluksiz bo`lsa, u holda uni shu *kesmada uzluksiz funksiya* deyiladi.

Kesmada uzluksiz funksiyaning xossalari *global xossalar*, nuqtada uzluksiz funksiyaning xossalari *lokal xossalar* deyiladi.

Kesmada uzluksiz funksiyaing quyidagi ajoyib xossasi aslida teskari funksiya mavjudligi alomati hisoblanadi:

Agar $f(x)$ funksiya $[a, b]$ kesmada uzluksiz va qat'iy monoton bo`lsa, u holda unga teskari funsiya mavjud va bu teskari funksiya uzluksiz hamda monoton bo`ladi.

$f(x)$ funksiya $[a,\ b]$ kesmada uzluksiz bo`lsin. U holda quyidagi tasdiqlar o`rinli:

1) $f(x)$ funksiya $[a,\ b]$ kesmada o`zining eng katta va eng kichik qiymatiga erishadi, ya'ni

$$\exists c_1 \in [a,\ b], f(c_1) = m = \min\{f(x):\ x \in [a,\ b]\},$$
$$\exists c_2 \in [a,\ b], f(c_2) = M = \max\{f(x):\ x \in [a,\ b]\},$$
$$m \le f(x) \le M, \forall x \in [a,\ b];$$

2) $f(x)$ funksiya $[m,\ M]$ kesmadagi ixtiyoriy qiymatga erishadi, ya'ni

$$\forall \mu, m \le \mu \le M, \exists c \in [a,\ b] \Rightarrow \mu = f(c).$$

Boshqacha aytganda, $f([a, b]) = [m,\ M]$.

3) $f(x)$ funksiya $[a,\ b]$ kesmaning oxirlarida turlicha ishorali qiymatlar qabul qilsa, ya'ni $f(a)f(b) < 0$ bo`lsa, u holda $(a,\ b)$ ichida shunday $c_0$ nuqta topiladiki, $f(c_0) = 0$ bo`ladi.

Bu xossalarning illyustratsiyalari yuqoridagi rasmda keltirilgan.

10.4. Funksiyaning uzilish nuqtalari va ularning tasniflanishi

Funksiya uzluksiz bo`lmaydigan nuqtalar funksiyaning *uzilish nuqtalari* deyiladi.

10.4.1.-misol. $f(x) = \operatorname{sign} x = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0 \end{cases}$ funksiya $x = 0$ nuqtada qiymatlari mos ravishda -1 va 1 bo`lgan chap tomonli limitga ham, o`ng tomonli limitga ham ega. $x = 0$ nuqta funksiyaning uzilish nuqtasi bo`ladi, chunki $f(0) = 0$ bo`lib, bu qiymat chap tomonli limitga ham, o`ng tomonli limitga ham teng emas.

Funksiya uzilish nuqtalarining tasniflanishi

I. Bartaraf etiladigan uzilish. Agar a nuqtada $f(x)$ funksiyaning $\lim\limits_{x\to a} f(x)$ limiti mavjud bo`lib, yoki bu nuqtada $f(x)$ ning qiymati aniqlanmagan, yoki uning $f(a)$ qiymati $\lim\limits_{x\to a} f(x)$ limitga teng bo`lmasa, a nuqta $f(x)$ funksiyaning *bartaraf etiladigan uzilish* nuqtasi deyiladi.

10.4.2.-misol. $f(x) = \frac{\sin x}{x}$ funksiya $x = 0$ nuqtada 1 ga teng limitga ega (1-ajoyib limit). Biroq, bu nuqta $x = 0$ ning o`zida mazkur funksiya aniqlanmagan. Demak, $x = 0$ nuqta bu funksiya uchun bartaraf etiladigan uzilish nuqtasi bo`ladi.

Funksiyani $x = 0$ da funksiya limiti qiymatiga teng deb aniqlab, ya'ni yangi

$$f_1(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0 \end{cases}$$

funksiya kiritib, bu uzilishni bartaraf etish mumkin. $f_1(x)$ funksiya butun son o`qida uzluksiz bo`ladi. Uzilish bartaraf etildi.

II. Birinchi tur uzilish. Agar a nuqtada chekli chap va o`ng limitlar mavjud bo`lib, ular bir-biriga teng bo`lmasa, ya'ni

$$\lim_{x\to a-0} f(x) \neq \lim_{x\to a+0} f(x)$$

bo`lsa, a nuqta $f(x)$ funksiya uchun *birinchi tur uzilish* nuqtasi deyiladi.

Bunda $\omega(x) = \lim\limits_{x\to a+0} f(x) - \lim\limits_{x\to a-0} f(x)$ ayirma $f(x)$ funksiya a nuqtadagi *sakrashi* deyiladi.

10.4.3.-misol. Ushbu

$$f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0 \end{cases}$$

funksiya uchun $x = 0$ nuqta birinchi tur uzilish nuqtasi bo`ladi.

III. Ikkinchi tur uzilish. Agar a nuqtada $f(x)$ funksiya bir tomonlama limitlardan hech bo`lmaganda bittasiga ega bo`lmasa, yoki hech bo`lmaganda bittasi cheksiz bo`lsa, u holda a nuqta $f(x)$ funksiayning *ikkinchi tur uzilish nuqtasi* deyiladi.

10.4.4.-misol. $f(x)=\frac{1}{x-2}$ funksiya uchun $x=2$ nuqta 2-tur uzilish nuqtasi bo`ladi, chunki $\lim\limits_{x\to 2+0}\frac{1}{x-2}=+\infty$, $\lim\limits_{x\to 2-0}\frac{1}{x-2}=-\infty$.

10.4.5.-misol. $f(x)=\sin\frac{1}{x}$ funksiya uchun $x=0$ nuqta 2-tur uzilish nuqtasi bo`ladi, chunki uning bu nuqtada chap tomonli limiti ham, o`ng tomonli limiti ham mavjud emas.

10.4.6.-misol. $f(x)=\mathrm{e}^{\frac{1}{x}}$ funksiya uchun $x=0$ nuqta 2-tur uzilish nuqtasi bo`ladi, chunki uning bu nuqtada chap tomonli limiti 0 ga teng $\lim\limits_{x\to 0-}\mathrm{e}^{\frac{1}{x}}=0$, o`ng tomonli limiti esa cheksiz bo`ladi $\lim\limits_{x\to 0+}\mathrm{e}^{\frac{1}{x}}=+\infty$.

Mustahkamlash uchun mashqlar

Uzluksiz funksiyalarga oid masalalar

1. Quyidagi funksiyalarning uzilish nuqtalari va ularning turlarini aniqlang. Grafiklarini yasang:

1. $f(x)=\begin{cases} x+4, & x<-1,\\ x^2+2, & -1\le x<1,\\ 2x, & x\ge 1.\end{cases}$

2. $f(x)=\begin{cases} x+1, & x\le 0,\\ (x+1)^2, & 0\le x\le 2,\\ -x+4, & x>2.\end{cases}$

3. $f(x)=\begin{cases} x+2, & x<-1,\\ x^2+1, & -1<x\le 1,\\ -x+3, & x>1.\end{cases}$

4. $f(x)=\begin{cases} -x, & x\le 0,\\ -(x-1)^2, & 0<x<1,\\ x-3, & x\ge 2.\end{cases}$

5. $f(x)=\begin{cases} -2(x+1), & x\le -1,\\ (x+1)^3, & -1<x<0,\\ x, & x\ge 0.\end{cases}$

6. $f(x)=\begin{cases} -x, & x\le 0,\\ x^2, & 0<x\le 2,\\ x+1, & x>2.\end{cases}$

7. $f(x)=\begin{cases} x^2+1, & x\le 1,\\ 2x, & 1<x\le 3,\\ x+2, & x>3.\end{cases}$

8. $f(x)=\begin{cases} x-3, & x<0,\\ x+1, & 0\le x\le 4,\\ 3+x, & x>4.\end{cases}$

9. $f(x)=\begin{cases} \sqrt{1-x}, & x\le 0,\\ 0, & 0<x\le 2,\\ x-2, & x>2.\end{cases}$

10. $f(x)=\begin{cases} 2x^2, & x\le 0,\\ x, & 0<x\le 1,\\ 2+x, & x>1.\end{cases}$

2. Ko`rsatlgan nuqtalarda funksiyalarni uzluksizlikka tekshiring.

1. $f(x)=2^{1/(x-3)}+1; x_1=3,\ x_2=4.$
2. $f(x)=5^{1/(x-3)}-1; x_1=3,\ x_2=4.$
3. $f(x)=(x+7)/(x-2); x_1=2,\ x_2=3.$
4. $f(x)=(x-5)/(x+3); x_1=-2,\ x_2=-3.$
5. $f(x)=4^{1/(3-x)}+2; x_1=2,\ x_2=3.$
6. $f(x)=9^{1/(2-x)}; x_1=0,\ x_2=2.$
7. $f(x)=2^{1/(x-5)}+1; x_1=4,\ x_2=5.$

8. $f(x) = 5^{1/(x-4)} - 2; x_1 = 3,\ x_2 = 4.$
9. $f(x) = 6^{1/(x-3)} + 3; x_1 = 3,\ x_2 = 4.$
10. $f(x) = 7^{1/(5-x)} + 1; x_1 = 4,\ x_2 = 5.$
11. $f(x) = (x-3)(x+4); x_1 = -5,\ x_2 = -4.$
12. $f(x) = (x+5)/(x-2); x_1 = 3,\ x_2 = 2.$
13. $f(x) = 5^{2/(x-3)}; x_1 = 3,\ x_2 = 4.$
14. $f(x) = 4^{2/(x-1)} - 3; x_1 = 1,\ x_2 = 2.$
15. $f(x) = 2^{5/(1-x)} - 1; x_1 = 0,\ x_2 = 1.$
16. $f(x) = 8^{4/(x-2)} - 1; x_1 = 2,\ x_2 = 3.$
17. $f(x) = 5^{4/(3-x)} + 1; x_1 = 2,\ x_2 = 3.$
18. $f(x) = 3x/(x-4); x_1 = 4,\ x_2 = 5.$

O`z-o`zini tekshirish uchun savollar

1. Funksiyaning nuqtadagi uzluksizligi deb nimaga aytiladi?
2. Asosiy elementar funksiyalarning uzluksizligini tushuntiring.
3. Kesmada uzluksiz funksiyalarning asosiy xossalarini tushuntiring.
4. Funksiyaning uzilish nuqtalari va ularning tasniflanishini tushuntiring.
5. Funksiya uzilish nuqtalarining tasniflanishini tushuntiring.

O`z-o`zini tekshirish uchun testlar

Uzluksiz funksiyalarga oid testlar

1. $y = f(x)$ funksiya ξ nuqtada uzluksizligining ta'rifini ko`rsating.

a) Ixtiyoriy $\varepsilon > 0$ sonni olganda ham, shunday $\delta > 0$ son topilib, $|x - \xi| < \delta$ tengsizlikni qanoatlantiruvchi har bir x nuqta uchun $|f(x) - f(\xi)| < \varepsilon$ tengsizlikning bajarilishi kelib chiqaversa, u holda $y = f(x)$ funksiya ξ nuqtada uzluksiz deyiladi.

b) Ixtiyoriy $\varepsilon > 0$ sonni olganda ham, shunday $\delta > 0$ son topilib, $|x - \xi| > \delta$ tengsizlikni qanoatlantiruvchi har bir x nuqta uchun $|f(x) - f(\xi)| > \varepsilon$ tengsizlikning bajarilishi kelib chiqaversa, u holda $y = f(x)$ funksiya ξ nuqtada uzluksiz deyiladi.

c) Ixtiyoriy $\varepsilon > 0$ sonni olganda ham, shunday $\delta > 0$ son topilib, $|x + \xi| < \delta$ tengsizlikni qanoatlantiruvchi har bir x nuqta uchun $|f(x) + f(\xi)| < \varepsilon$ tengsizlikning bajarilishi kelib chiqaversa, u holda $y = f(x)$ funksiya ξ nuqtada uzluksiz deyiladi.

d) Ixtiyoriy $\varepsilon > 0$ sonni olganda ham, shunday $\delta > 0$ son topilib, $|x - \xi| < \delta$ tengsizlikni qanoatlantiruvchi har bir x nuqta uchun $|f(x) + f(\xi)| < \varepsilon$ tengsizlikning bajarilishi kelib chiqaversa, u holda $y = f(x)$ funksiya ξ nuqtada uzluksiz deyiladi.

2. $y = f(x)$ funksiya ξ nuqtada uzluksizligining ketma-ketlik orqali berilgan ta'rifini ko`rsating.

a) ξ nuqtaga yaqinlashuvchi ixtiyoriy $\{x_n\} \subset X$ ketma-ketlikni olganda ham, mos qiymatlardan tuzilgan $\{f(x_n)\}$ ketma-ketlik $f(\xi)$ qiymatga intilaversa, u holda $y = f(x)$ funksiya ξ nuqtada uzluksiz deyiladi.

b) ξ nuqtaga yaqinlashuvchi ixtiyoriy $\{x_n\} \subset X$ ketma-ketlikni olganda ham, mos qiymatlardan tuzilgan $\{f(x_n)\}$ ketma-ketlik $f(\xi)$ qiymatdan uzoqlashsa, u holda $y = f(x)$ funksiya ξ nuqtada uzluksiz deyiladi.

c) ξ nuqtaga yaqinlashuvchi ixtiyoriy $\{x_n\} \subset X$ ketma-ketlikni olganda ham, mos qiymatlardan tuzilgan $\{f(x_n)\}$ ketma-ketlik ξ ga teng bo`lsa, u holda $y = f(x)$ funksiya ξ nuqtada uzluksiz deyiladi.

d) bunday ta'rif yo`q.

3. Agar ... bo`lsa, u holda $y = f(x)$ funksiya ξ nuqtada uzluksiz deyiladi.

a) $\lim\limits_{x\to\xi} f(x) = f(\xi)$ b) $\lim\limits_{x\to\xi} f(x) \neq f(\xi)$

c) $\lim\limits_{x\to\xi} f(x) = 0$ d) $\lim\limits_{x\to\xi} f(x) = 1$

4. $y = f(x)$ funksiya ξ nuqtada uzluksiz bo`lsa, u holda ... bo`ladi.

a) $\lim\limits_{x\to\xi} f(x) = f\left(\lim\limits_{x\to\xi} x\right)$ b) $\lim\limits_{x\to\xi} f(x) \neq f\left(\lim\limits_{x\to\xi} x\right)$

c) $\lim\limits_{x\to\xi} f(x) < f\left(\lim\limits_{x\to\xi} x\right)$ d) $\lim\limits_{x\to\xi} f(x) > f\left(\lim\limits_{x\to\xi} x\right)$

5. Agar ... bo`lsa, $f(x)$ funksiya $\xi$ nuqtada o`ngdan uzluksiz deyiladi.

a) $\lim\limits_{x\to\xi-0} f(x) = f(\xi)$ b) $\lim\limits_{x\to\xi-0} f(x) > f(\xi)$

c) $\lim\limits_{x\to\xi+0} f(x) < f(\xi)$ d) $\lim\limits_{x\to\xi+0} f(x) = f(\xi)$

6. Agar ... bo`lsa, $f(x)$ funksiya ξ nuqtada chapdan uzluksiz deyiladi.

a) $\lim\limits_{x\to\xi-0} f(x) = f(\xi)$ b) $\lim\limits_{x\to\xi+0} f(x) > f(\xi)$

c) $\lim\limits_{x\to\xi-0} f(x) < f(\xi)$ d) $\lim\limits_{x\to\xi+0} f(x) = f(\xi)$

7. $f(x)$ va $g(x)$ funksiyalar a nuqtada uzluksiz bo`lsa, u holda $f(x) \pm g(x)$ funksiya ham a nuqtada ... .

a) uzluksiz bo`ladi b) uzluksiz bo`lmaydi

c) qiymati 1 teng bo`ladi d) qiymati 0 teng bo`ladi

8. $f(x)$ va $g(x)$ funksiyalar a nuqtada uzluksiz bo`lsa, u holda $f(x)g(x)$ funksiya ham a nuqtada ... .

a) uzluksiz bo`ladi b) uzluksiz bo`lmaydi

c) qiymati 1 teng bo`ladi d) qiymati 0 teng bo`ladi

9. $z = \varphi(x)$ funksiya x_0 nuqtada uzluksiz, $y = f(z)$ funksiya $z_0 = \varphi(x_0)$ nuqtada uzluksiz bo`lsa, $y = f(\varphi(x))$ murakkab funksiya x_0 nuqtada ... .

a) uzluksiz bo`ladi b) uzluksiz bo`lmaydi

c) qiymati 1 teng bo`ladi d) qiymati 0 teng bo`ladi

10. $f(x)$ funksiya $[a,\ b]$ kesmaning har bir nuqtasida uzluksiz bo`lsa, u holda uni shu ... deyiladi.

a) kesmada uzluksiz funksiya b) kesmada uzilish ega funksiya

c) kesmada o`zgarmas funksiya d) kesmada aniqlanmagan funksiya

11. Kesmada uzluksiz funksiyaning xossalari ... deyiladi

a) global xossalar b) lokal xossalar

c) eng katta xossalar d) eng kichik xossalar

12. Nuqtada uzluksiz funksiyaning xossalari ... deyiladi.

a) global xossalar b) lokal xossalar

c) eng katta xossalar d) eng kichik xossalar

13. Agar $f(x)$ funksiya $[a, b]$ kesmada uzluksiz va qat'iy monoton bo`lsa, u holda unga ... .

a) teskari funsiya mavjud va bu teskari funksiya uzluksiz hamda monoton bo`ladi.

b) teskari funsiya mavjud emas

c) teskari funsiya mavjud va bu teskari funksiya uzluksiz emas.

d) teskari funsiya mavjud va bu teskari funksiya uzilish ega hamda monoton bo`ladi.

14. $f(x)$ funksiya $[a,\ b]$ kesmada uzluksiz bo`lsa, u holda ... .

a) $f(x)$ funksiya $[a,\ b]$ kesmada o`zining eng katta va eng kichik qiymatiga erishadi

b) $f(x)$ funksiya $[a,\ b]$ kesmada o`zining faqat eng katta qiymatiga erishadi

c) $f(x)$ funksiya $[a,\ b]$ kesmada o`zining faqat eng kichik qiymatiga erishadi

d) $f(x)$ funksiya $[a,\ b]$ kesmada o`zining eng katta yoki eng kichik qiymatiga erishadi

15. Funksiya uzluksiz bo`lmaydigan nuqtalar funksiyaning ... deyiladi.

a) uzilish nuqtalari b) uzluksiz nuqtalari

c) limit nuqtalari d) sakrash nuqtalari

16. Agar a nuqtada $f(x)$ funksiyaning $\lim\limits_{x\to a} f(x)$ limiti mavjud bo`lib, yoki bu nuqtada $f(x)$ ning qiymati aniqlanmagan, yoki uning $f(a)$ qiymati $\lim\limits_{x\to a} f(x)$ limitga teng bo`lmasa, a nuqta $f(x)$ funksiyaning ... nuqtasi deyiladi.

a) bartaraf etiladigan uzilish b) birinchi tur uzilish

c) ikkinchi tur uzilish d) sakrash

17. Agar a nuqtada chekli chap va o`ng limitlar mavjud bo`lib, ular bir-biriga teng bo`lmasa, ya'ni $\lim\limits_{x\to a-0} f(x) \neq \lim\limits_{x\to a+0} f(x)$ bo`lsa, a nuqta $f(x)$ funksiya uchun ... nuqtasi deyiladi.

a) bartaraf etiladigan uzilish b) birinchi tur uzilish

c) ikkinchi tur uzilish d) sakrash

18. Agar a nuqtada $f(x)$ funksiya bir tomonlama limitlardan hech bo`lmaganda bittasiga ega bo`lmasa, yoki hech bo`lmaganda bittasi cheksiz bo`lsa, u holda a nuqta $f(x)$ funksiayning ...nuqtasi deyiladi.

a) bartaraf etiladigan uzilish b) birinchi tur uzilish

c) ikkinchi tur uzilish d) sakrash

19. $\omega(x) = \lim\limits_{x\to a+0} f(x) - \lim\limits_{x\to a-0} f(x)$ ayirma $f(x)$ funksiya a nuqtadagi ... deyiladi.

a) bartaraf etiladigan uzilishi b) birinchi tur uzilishi

c) ikkinchi tur uzilishi d) sakrashi

20. $f(x) = \frac{\sin x}{x}$ funksiyaning uzilish nuqtasi toping.

a) $x = 0$ b) $x = 1$ c) $x = -1$ d) $x = \frac{\pi}{2}$

21. $f(x) = \frac{\sin x}{x}$ funksiya qanday uzilish nuqtasi ega.

a) bartaraf etiladigan uzilishi nuqtasi ega
b) birinchi tur uzilishi uzilishi nuqtasi ega
c) ikkinchi tur uzilishi uzilishi nuqtasi ega
d) uzilishi nuqtasiga ega emas.

22. $f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0 \end{cases}$ funksiyaning uzilish nuqtasi toping.

a) $x = 0$ b) $x = 1$ c) $x = -1$ d) $x = \frac{1}{2}$

23. $f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0 \end{cases}$ funksiya qanday uzilish nuqtasi ega.

a) bartaraf etiladigan uzilishi nuqtasi ega
b) birinchi tur uzilishi uzilishi nuqtasi ega
c) ikkinchi tur uzilishi uzilishi nuqtasi ega
d) uzilishi nuqtasiga ega emas.

24. $f(x) = \frac{1}{x-2}$ funksiyaning uzilish nuqtasi toping.

a) $x = 0$ b) $x = 2$ c) $x = -1$ d) $x = \frac{1}{2}$

25. $f(x) = \frac{1}{x-2}$ funksiya qanday uzilish nuqtasi ega.

a) bartaraf etiladigan uzilishi nuqtasi ega
b) birinchi tur uzilishi uzilishi nuqtasi ega
c) ikkinchi tur uzilishi uzilishi nuqtasi ega
d) uzilishi nuqtasiga ega emas.

26. $f(x) = \sin\frac{1}{x}$ funksiyaning uzilish nuqtasi toping.

a) $x = 0$ b) $x = 2$ c) $x = -1$ d) $x = \frac{1}{2}$

27. $f(x) = \sin\frac{1}{x}$ funksiya qanday uzilish nuqtasi ega.

a) bartaraf etiladigan uzilishi nuqtasi ega
b) birinchi tur uzilishi uzilishi nuqtasi ega
c) ikkinchi tur uzilishi uzilishi nuqtasi ega
d) uzilishi nuqtasiga ega emas.

28. $f(x) = \frac{1}{1-x}$ funksiyaning uzilish nuqtasi toping.

a) $x = 0$ b) $x = 2$ c) $x = 1$ d) $x = \frac{1}{2}$

29. $f(x) = \frac{1}{1-x}$ funksiya qanday uzilish nuqtasi ega.

a) bartaraf etiladigan uzilishi nuqtasi ega
b) birinchi tur uzilishi uzilishi nuqtasi ega
c) ikkinchi tur uzilishi uzilishi nuqtasi ega

d) uzilishi nuqtasiga ega emas.

30. $f(x) = \frac{x+3}{2-x}$ funksiyaning uzilish nuqtasi toping.

a) $x = 0$ b) $x = 2$ c) $x = 1$ d) $x = \frac{1}{2}$

11-§. HOSILA

11.1. Funksiya hosilasi (nazariy qism)

$f(x)$ funksiya X oraliqda aniqlangan bo`lsin. Argumentning $x_0$ qiymatiga ixtiyoriy $\Delta x$ *orttirma*ni shunday beramizki, $x_0 + \Delta x$ nuqta ham $X$ oraliqqa tegishli bo`lsin. Orttirma musbat yoki manfiy bo`lishi mumkin, lekin har doim noldan farqli ($\Delta x \neq 0$) bo`ladi. Ushbu $f(x_0 + \Delta x) - f(x_0)$ ayirma $y = f(x)$ funsiyaning x_0 nuqtadagi *orttirmasi* deyiladi va Δy orqali belgilanadi. Demak,

$$\Delta y = f(x_0 + \Delta x) - f(x_0).$$

Agar $\frac{\Delta y}{\Delta x}$ nisbatning $\Delta x \to 0$ da limiti mavjud bo`lsa, u holda bu limit $y = f(x)$ funsiyaning x_0 nuqtadagi *hosilasi* deb ataladi va $y'(x_0)$ yoki $f'(x_0)$ orqali belgilanadi. Demak,

$$y'(x_0) = f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Funksiya hosilasini topish amali *funksiyani differensiallash* deb yuritiladi; nuqtada hosilaga ega funksiya shu nuqtada *differensiallanuvchi* deyiladi.

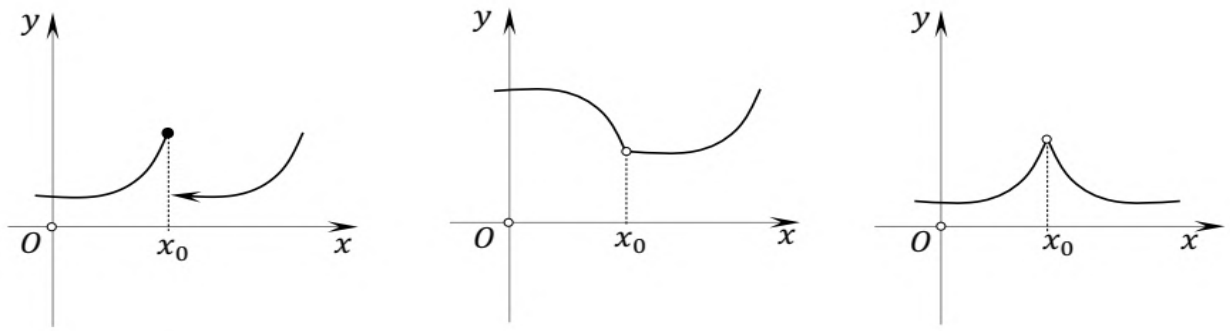
$\lim\limits_{\Delta x \to 0+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ limit mavjud bo`lsa, bu limit $y = f(x)$ funksiyaning $x_0$ nuqtadagi *o`ng tomonlama* hosilasi deyiladi va quyidagi kabi belgilanadi:

$$y'_+(x_0) = f'_+(x_0) = \lim_{\Delta x \to 0+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x};$$

shunga o`xshash *chap tomonlama hosila* ham aniqlanadi:

$$y'_-(x_0) = f'_-(x_0) = \lim_{\Delta x \to 0-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Funksiyaning o`ng va chap tomonlama hosilalari *bir tomonlama hosilalar* deb yuritiladi. Nuqtada differensiallanuvchi funksiya ustma-ust tushadigan bir tomonlama hosilalarga ega bo`ladi. Agar funksiyaning nuqtadagi bir tomonlama hosilalari mavjud bo`lib, ular ustma-ust tushsa, u holda bu funsiya shu nuqtada differensiallanuvchi bo`ladi. Nuqtadagi bir tomonlama hosilalari mavjud bo`lib, ular bir-biriga teng bo`lmaydigan funksiyalar mavjud. Masalan, $f(x) = |x|$ funksiya $x = 0$ nuqtada o`zaro teng bo`lmaydigan bir tomonlama hosilalarga ega: $f'_-(0) = -1$, $f'_+(0) = 1$.



Nuqtada differensiallanuvchi funksiya shu nuqtada uzluksiz bo`ladi. Aksinchasi har doim ham o`rinli bo`lavermaydi.Masalan, $f(x) = |x|$ funksiya $x = 0$ nuqtada uzluksiz, lekin differensiallanuvchi emas. Funksiya uzluksiz bo`ladigan, lekin differensiallanuvchi bo`lmaydigan nuqta funksiya grafigining *burchagi*, yoki *sinish nuqtasi* deyiladi.

Agar biror x_0 nuqtada $\frac{\Delta y}{\Delta x}$ nisbatning limiti cheksiz, ya'ni

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = +\infty \text{ yoki } \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = -\infty$$

bo`lsa, u holda $f(x)$ funksiya x_0 nuqtada *cheksiz hosilaga ega* deyiladi.

11.1.1.-misol. $f(x) = \sqrt{x}$ funksiyaning $x = 0$ nuqtadagi hosilasini toping.

Yechish. Berilgan funksiya uchun x o`zgaruvchining qabul qiladigan qiymatlari to`plami $[0, +\infty)$ ekanligidan Δx orttirma musbat bo`lishi kelib chiqadi. Ta'rifga ko`ra

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sqrt{x_0+\Delta x}-\sqrt{x_0}}{\Delta x} =$$
$$= \lim_{\Delta x\to 0}\frac{\sqrt{0+\Delta x}-\sqrt{0}}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sqrt{\Delta x}}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\sqrt{\Delta x}} = +\infty.$$

Shunday qilib, $\sqrt{x}$ funksiya $x = 0$ nuqtada cheksiz hosilaga ega ekan.

11.1.2.-misol. $f(x) = \sqrt{x}$ funksiyaning x nuqtadagi hosilasini toping, $x > 0$.

Yechish. Hosila ta'rifidan foydalanamiz

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} =$$
$$= \lim_{\Delta x\to 0}\frac{(\sqrt{x+\Delta x}-\sqrt{x})(\sqrt{x+\Delta x}+\sqrt{x})}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})} = \lim_{\Delta x\to 0}\frac{x+\Delta x-x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})} =$$
$$= \lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Shu bilan $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$ tenglik isbotlandi.

Quyidagi tengliklarni isbotlash mustaqil ishlash uchun taqdim etiladi. Bu tengliklar jamlanmasi *asosiy elementar funksiyalarning hosilalari jadvali* (qisqacha, *hosilalar jadvali*) deb yuritiladi.

Hosilalar jadvali

1. $C' = 0$, bunda C – o`zgarmas funksiya.
2. $(x^\alpha)' = \alpha x^{\alpha-1}$; xususan, $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$, $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$.
3. $(\log_a x)' = \frac{1}{x\ln a}$; xususan, $(\ln x)' = \frac{1}{x}$.
4. $(a^x)' = a^x \ln a$; xususan, $(\mathrm{e}^x)' = \mathrm{e}^x$.
5. $(\sin x)' = \cos x$.
6. $(\cos x)' = -\sin x$.
7. $(\mathrm{tg}\, x)' = \frac{1}{\cos^2 x}$.
8. $(\mathrm{ctg}\, x)' = -\frac{1}{\sin^2 x}$.
9. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.
10. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$.

11. $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$.

12. $(\operatorname{arcctg} x)' = -\frac{1}{1+x^2}$.

11.2. Funksiyalarning differensiallanuvchanligi

x nuqtadagi orttirmasi

$$\Delta y = A(x)\Delta x + o(\Delta x)\Delta x$$

singari ifodalanishi mumkin bo`lgan $y = f(x)$ funksiya $x$ nuqtada *differensiallanuvchi funksiya* deyiladi. Bu tenglikning o`ng tomonidagi asosiy chiziqli $A(x)\Delta x$ hadi $y = f(x)$ *funksiyaning differensiali* deyiladi va $df(x)$ yoki dy orqali belgilanadi. Demak, ta'rifga ko`ra, $df(x) = A(x)\Delta x$ yoki $dy = A(x)\Delta x$.

Ushbu muhim

$$\boldsymbol{dx = \Delta x}$$

tenglikni eslab qoling. Uni isbotlash uchun $y = x$ funksiya qaraladi. Ravshanki, $\Delta y = \Delta x = A(x)\Delta x + o(\Delta x)\Delta x$. Bu tenglikni Δx ga bo`lib,

$$\frac{\Delta y}{\Delta x} = 1 = A(x) + o(\Delta x)$$

ko`rinishga keltiramiz. $\lim\limits_{\Delta x \to 0} o(\Delta x) = 0$ bo`lgani uchun

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1 = \lim_{\Delta x \to 0} \big(A(x) + o(\Delta x)\big) = A(x),$$

ya'ni $A(x) = 1$ ni hosil qilamiz. Endi $dy = A(x)\Delta x$ tenglikni $y = x$ funksiya uchun qayta yozamiz: $dy = dx = A(x)\Delta x = \Delta x$, ya'ni $dx = \Delta x$.

$y = f(x)$ funksiyaning x nuqtada differensiali mavjud bo`lishi uchun $y' = f'(x)$ hosila chekli bo`lishi zarur va yetarli. Bunda $f'(x) = A(x)$ bo`ladi. Endi funksiya differensialini quyidagicha yozish mumkin:

$$df(x) = f'(x)dx \text{ yoki } dy = f'(x)dx.$$

Yuqorida bayon qilinganlarni ushbu misolda ko`rsatamiz: $f(x) = \sqrt{x}$ funksiya $x = 0$ nuqtada differensiallanuvchi emas, hamda barcha $x > 0$ nuqtalarda $d\sqrt{x} = \frac{1}{\sqrt{x}}dx$ bo`ladi.

Funksiya differensiali dy va orttirmasi Δy uchun $dy = \Delta y$ tenglik bajarilishi shart emas.

11.2.1.-misol. $y = 3x^3 + x - 1$ funksiyaning $x = 1$ nuqtada $\Delta x = 0{,}1$ bo`lgandagi funksiya orttirmasini va differensialini toping. Funksiya orttirmasini uning differensiali bilan almashtirgandagi absolyut va nisbiy xatoliklarni toping.

Yechish. Berilgan funksiya uchun funksiya orttirmasi ta'rifidan, ya'ni

$$\Delta y = f(x + \Delta x) - f(x),$$

tenglikdan

$$\Delta y = (3(x + \Delta x)^3 + (x + \Delta x) - 1) - (3x^3 + x - 1) =$$
$$= 3x^3 + 9x^2\Delta x + 9x\Delta x^2 + 3\Delta x^3 + x + \Delta x - 1 - 3x^3 - x + 1 =$$
$$= 9x^2\Delta x + 9x\Delta x^2 + 3\Delta x^3 + \Delta x,$$

ya'ni

$$\Delta y = 9x^2\Delta x + 9x\Delta x^2 + 3\Delta x^3 + \Delta x$$

orttirmaga ega bo`lamiz. $x = 1$ va $\Delta x = 0{,}1$ qiymatlarni qo`yib,

$$\Delta y = 0{,}9 + 0{,}09 + 0{,}003 + 0{,}1 = 1{,}093$$

funksiya orttirmasiga ega bo`lamiz.

Endi funksiya differensialini topaylik. Ta'rifga ko`ra

$$dy = A(x)\Delta x = f'(x)\Delta x = (3x^3 + x - 1)'\Delta x = (9x^2 + 1)\Delta x,$$

ya'ni

$$dy = (9x^2 + 1)\Delta x.$$

$x = 1$ va $\Delta x = 0{,}1$ qiymatlarni qo`ysak,

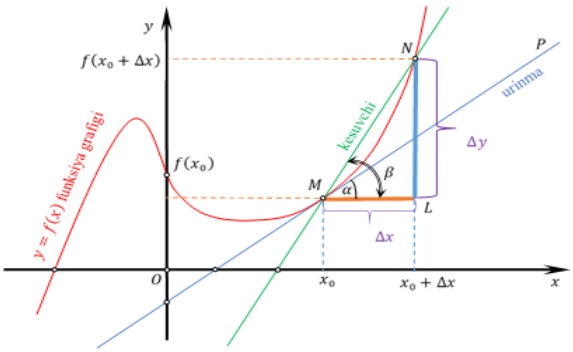
$$dy\Big|_{\substack{x=1,\\ \Delta x=0{,}1}} = (9x^2 + 1)\Delta x\Big|_{\substack{x=1,\\ \Delta x=0{,}1}} = (9 + 1)\cdot 0{,}1 = 1.$$

ya'ni $dy = 1$ hosil bo`ladi.

Funksiya orttirmasini uning differensiali bilan almashtirgandagi absolyut xatolik $|\Delta y - dy| = 0{,}093$ va nisbiy xatolik $\left|\frac{\Delta y - dy}{\Delta y}\right| = \frac{0{,}093}{1{,}093} \approx 0{,}085$ yoki 8,5% bo`ladi.

11.3. Hosilaning geometrik va mexanik ma'nosi. Differensialning geometrik ma'nosi

x_0 nuqtaning biror atrofida differensiallanuvchi bo`lgan $y = f(x)$ funksiyani qaraylik. Argumentga x_0 nuqtada Δx orttirma beramiz. Funksiya uzluksizligidan argumentning x_0 nuqtadagi cheksiz kichik Δx orttirmasiga $y = f(x)$ funksiyaning cheksiz kichik $\Delta y = f(x_0 + \Delta x) - f(x_0)$ orttirmasi mos keladi.



Mavzuning keyingi bayonida egri chiziq deganda $y = f(x)$ funksiyaning grafigi tushuniladi.

Δx kattalik kichiklashib, 0 ga cheksiz yaqinlashganda N nuqta egri chiziq bo`ylab $M$ nuqtaga cheksiz yaqinlashadi. $MN$ kesuvchi esa $M$ nuqta atrofida buriladi. $\Delta x$ kattalik kichiklashib, 0 ga cheksiz yaqinlashganda $MN$ kesuvchining limit holati egri chiziqqa $M$ nuqtada o`tkazilgan *urinma* deyiladi. Odatda bu bu limit holat $y = f(x)$ funksiya grafigiga x_0 nuqtada o`tkazilga urinma deb yuritiladi.

Rasmdan ko`rinib turibdiki, $\operatorname{tg}\beta = \frac{\Delta y}{\Delta x}$, bu yerda $\beta$ – absissa o`qining musbat yo`nalshi va MN kesuvchi orasidagi burchak. Ravshanki,

$$\operatorname{tg}\alpha = \lim_{\Delta x \to 0} \operatorname{tg}\beta = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0).$$

Shunday qilib, x_0 nuqtada differensiallanuvchi $y = f(x)$ funksiya uchun

$$\boxed{tg\alpha = f'(x_0)}.$$

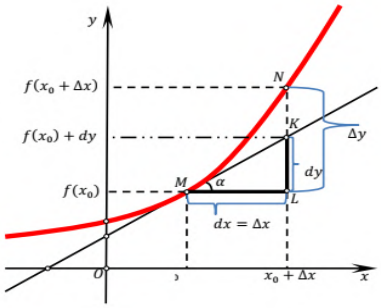
bo`ladi. Bu hosilaning *geometrik ma'nosini* ifodalaydi. Boshqacha aytganda, $f(x)$ funksiyaning x_0 nuqtadagi $f'(x_0)$ hosilasi **geometrik nuqtai nazardan** $y = f(x)$

egri chiziqning $(x_0,\ f(x_0))$ nuqtasidagi urinmasi bilan absissa o`qining musbat yo`nalshi orasidagi burchuk **tangensini** anglatadi.

Hosilaning **mexanik ma'nosi** qaralayotgan nuqtadagi funksiyaning o`zgarish tezligidan iborat.

Endi differensialning geometrik ma'nosini bayon etamiz.

x_0 nuqtada differensiallanuvchi $y = f(x)$ funksiya uchun ta'rifga ko`ra $dy = f'(x_0)dx$ bo`ladi. Hosilaning geometrik ma'nosiga ko`ra $\operatorname{tg}\alpha = f'(x_0)$. Demak, $dy = \operatorname{tg}\alpha dx$. Bundan, $dx = \Delta x$ bo`lgani uchun $dy = \operatorname{tg}\alpha\Delta x$ ga ega bo`lamiz.



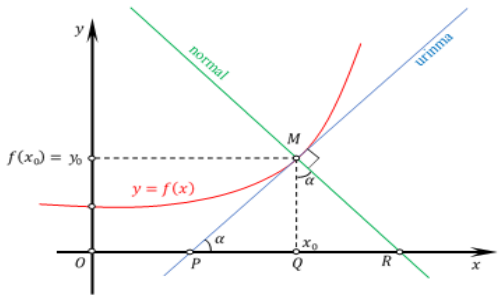
$y = f(x)$ egri chiziqning $M(x_0,\ f(x_0))$ nuqtasidagi MK urinmasini o`tkazib, urinmaning $x_0 + \Delta x$ nuqtadagi ordinatasini qaraymiz. Rasmdan bu ordinata $f(x_0) + dy$ ekanigi ko`rinib turibdi. Bu holat x_0 argument Δx orttirma olganda $y = f(x)$ funksiya grafigining $M(x_0,\ f(x_0))$ nuqtadagi urinmasi $dy = f'(x_0)dx = f'(x_0)\Delta x$ orttirma olishini anglatadi. Shunday qilib, berilgan nuqta orttirma olganda funksiya grafigining shu nuqtadagi urinmasi ordinatasining orttirmasi *differensialning geometrik ma'nosini* ifodalar ekan.

Differensialning mexanik ma'nosi biror qaralayotgan vaqt onidan boshlab erishgan tezligi bilan tekis harakatlanayotgan moddiy nuqta yo`lining differensiali shu yo`l orttirmasiga teng ekanligidan iborat.

11.4. Funksiya grafigining berilgan nuqtadagi urinmasi va normali tenglamalari

$y = f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo`lsin.

$(x_0,\ y_0)$ nuqtadan o`tuvchi, burchak koeffisiyenti $k$ bo`lgan to`g`ri chiziq tenglamasi $y = y_0 + k(x - x_0)$ ko`rinishda bo`ladi. Lekin $k = \operatorname{tg}\alpha$ va $\operatorname{tg}\alpha = f'(x_0)$ tengliklardan $k = f'(x_0)$ bo`ladi.



$y = f(x)$ funksiya grafigining x_0 nuqtadagi ordinatasi $y_0 = f(x_0)$ ekanligidan yuqoridagi tenglama

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0)}$$

ko`rinishga keladi. Bu $y = f(x)$ funksiyaning x_0 nuqtadagi *urinma tenglamasidir*.

Egri chiziqning berilgan nuqtadagi *normali* deb, egri chiziqning shu nuqtadagi urinmasiga perpendikulyar bo`lgan to`g`ri chiziqqa aytiladi.

Burchak koffisiyentlari k_1 va k_2 bo`lgan to`g`ri chiziqlarning perpendikulyarlik sharti $k_2 = -\frac{1}{k_1}$ tenglikdan iborat. Bundan normalning burchak koeffisiyenti $-\frac{1}{f'(x_0)}$ ga tengligi kelib chiqadi. Shunday qilib, $y = f(x)$ funksiyaning $x_0$ nuqtadagi normali tenglamasi ushbu ko`rinishda bo`ladi

$$\boxed{y = f(x_0) - \frac{1}{f'(x_0)}(x - x_0)}.$$

$PQ = \frac{MQ}{\operatorname{tg}\alpha}$ kesma *urinmaosti*, $QR = MQ\operatorname{tg}\alpha$ kesma esa *normalosti* deyiladi. Bu kesmalar uchun ushbu tengliklar o`rinli

$$PQ = \frac{f(x_0)}{f'(x_0)} \text{ va } QR = f(x_0)f'(x_0),$$

$$|PQ| = \left|\frac{f(x_0)}{f'(x_0)}\right| \text{ va } |QR| = |f(x_0)f'(x_0)|.$$

11.5. Differensialning taqribiy hisoblashlarga tadbiqlari

Hosila ta'rifidagi, ya'ni

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

tenglikdagi limit belgisi tashlab yuborilsa, Δx ning tayin kichik qiymatida ushbu $f'(x_0) \approx \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ taqribiy tenglikka erishamiz. $df(x) = f'(x)dx$ va $dx = \Delta x$ tengliklar e'tiborga olinsa,

$$dy = f'(x_0)dx = f'(x_0)\Delta x \approx \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\Delta x = f(x_0 + \Delta x) - f(x_0) = \Delta y$$

munosabatlarga ega bo`lamiz. Bu yerdan quyidagilarni ajratib ko`rsatamiz:

1) $\boldsymbol{dy \approx \Delta y}$;

2) $f'(x_0)\Delta x \approx f(x_0 + \Delta x) - f(x_0)$ yoki

$$\boxed{\boldsymbol{f(x_0 + \Delta x) \approx f'(x_0)\Delta x + f(x_0)}}.$$

Ramka ichiga olingan formula taqribiy hisoblashlarda keng qo`llaniladi.

11.5.1.-misol. Differensial tushunchasidan foydalanib, $y = \sqrt[5]{\frac{2-x}{2+x}}$ funksiyaning $x = 0{,}15$ dagi taqribiy qiymatini toping.

Yechish. $f(x_0 + \Delta x) \approx f'(x_0)\Delta x + f(x_0)$ formulani qo`llash uchun bizdan:

1) argumentning x_0 qiymati,

2) Δx orttirma,

3) funksiyaning x_0 nuqtadagi $f(x_0)$ qiymati va

4) hosilaning x_0 nuqtadagi $f'(x_0)$ qiymati

talab etiladi.

$x_0 = 0$, $\Delta x = 0{,}15$ deb olaylik. U holda $f(x_0) = y(0) = \sqrt[5]{\frac{2-0}{2+0}} = \sqrt[5]{1} = 1.$

Hosilani topamiz

$$y' = \left(\sqrt[5]{\frac{2-x}{2+x}}\right)' = \frac{1}{5}\sqrt[5]{\left(\frac{2+x}{2-x}\right)^4} \cdot \frac{-4}{(2+x)^2} = -\frac{4}{5} \cdot \frac{1}{(2+x)^2} \cdot \sqrt[5]{\left(\frac{2+x}{2-x}\right)^4}$$

va uning $x_0 = 0$ nuqtadagi qiymatini hisoblaymiz:

$$f'(x_0) = y'(0) = -\frac{4}{5} \cdot \frac{1}{(2+0)^2} \cdot \sqrt[5]{\left(\frac{2+0}{2-0}\right)^4} = -\frac{1}{5} = -0{,}2.$$

Nihoyat,

$$f(x_0+\Delta x)=y(0+0{,}15)=y(0{,}15)=\sqrt[5]{\frac{2-0{,}15}{2+0{,}15}}\approx -0{,}2\cdot 0{,}15+1=0{,}97,$$

ya'ni $\sqrt[5]{\frac{1{,}85}{2{,}15}}\approx 0{,}97$.

Kalkulyatordagi hisoblar $\sqrt[5]{\frac{1{,}85}{2{,}15}}\approx 0{,}9704$ ni beradi, aniqlik 10^{-4} darjada ekan.

11.5.2.-misol. Quyidagi ifodalarning taqribiy qiymatlarini toping:

$a)\ \cos 31°$; $b)\ \sin 89°$; $c)\ \lg 10{,}21$; $d)\ \log_3 2{,}9$;

$e)\ \sqrt[5]{33}$; $f)\ \sqrt[3]{26}$; $g)\ ctg45°10'$; $h)\ tg59°30'$.

11.5.3.-Misol. Qirrasi 5 sm bo`lgan mis kubik barcha yoqlaridan bir xil silliqlandi. Uning massasi 0,96 g ga kamayganini bilib hamda mis zichligini 8 g/sm^3 deb olib, kubning qirralari qanchaga qisqarganini aniqlang.

Yechish. Qirrasi x bo`lgan kub hajmi $V=x^3$ bo`ladi. Hajmni $V=\frac{m}{\rho}$ orqali ifodalash mumkin; kub hajmi o`zgarishi $\Delta V=\frac{\Delta m}{\rho}=\frac{-0{,}96}{8}=-0{,}12$ sm$^3$. Taqribiy $\Delta V\approx dV$ tenglik, $dx=\Delta x$ tenglik va $dV=V'dx=(x^3)'dx=3x^2dx$ larni e'tiborga olib, $\Delta V\approx 3x^2\Delta x$ ga ega bo`lamiz. Natijada $-0{,}12\approx 3\cdot 5^2\cdot\Delta x$, va bu yerdan

$\Delta x\approx -\frac{0{,}12}{3\cdot 25}=-0{,}0016$ sm.

Javob: Kub qirrasi 0,0016 sm ga qisqargan.

Mustahkamlash uchun mashqlar

Hosilaga oid masalalar

1. Hosila ta'rifidan foydalanib, *y=f(x)* funksiyalar uchun *y'* hosilasini toping:

1) $y=\frac{1}{x}$ 2) $y=\sqrt{x}$

3) $y=\sin x$ 4) $y=\cos x$

2. Quyidagi funksiyalarning hosilasini toping:

1. $y=5x^4-3\sqrt[7]{x^3}+\frac{7}{x^5}+4$; 2. $y=x^3\sin x$;

3. $y=(x^4+1)/(x^4-1)$; 4. $y=(x^5+3x-1)^4$;

5. $y=\sqrt[3]{\left(\frac{x^3+1}{x^3-1}\right)^2}$.

3. Quyidagi funksiyalarning hosilasini toping:

1. $y=\sqrt[3]{3x^4+2x-5}+\frac{4}{(x-2)^5}$ 2. $y=\sqrt[3]{(x-3)^4}-\frac{3}{2x^3-3x+1}$

3. $y=\sqrt{(x-4)^5}+\frac{5}{(2x^2+4x-1)^2}$ 4. $y=\sqrt[5]{7x^2-3x+5}-\frac{5}{(x-1)^3}$

5. $y=\sqrt[4]{3x^2-x+5}-\frac{3}{(x-5)^4}$ 6. $y=\sqrt{3x^4-2x^3+x}-\frac{4}{(x+2)^3}$

7. $y=\sqrt[3]{(x-7)^5}+\frac{5}{4x^2+3x-5}$ 8. $y=\sqrt[5]{(x+4)^6}-\frac{2}{2x^2-3x+7}$

9. $y=\frac{3}{(x-4)^7}-\sqrt{5x^2-4x+3}$ 10. $y=\sqrt[3]{4x^2-3x-4}-\frac{2}{(x-3)^5}$

4. Quyidagi funksiyalarning differensialini toping:

1. $y = 3x^3 - 2x^2 + 3x - 1$
2. $y = 5x^4 - 3\sqrt[7]{x^3} + \frac{7}{x^5} + 4;$
3. $y = x^3 \sin x;$
4. $y = \frac{x^4+1}{(x^4-1}$;
5. $y = (x^5 + 3x - 1)^4;$
6. $y = \sqrt[3]{(\frac{x^3+1}{x^3-1})}$

5.

a) $y = x^3 + 2x — 2$ funksiyaga $x_0 = 1$ nuqtada o`tkazilgan urinma va normal tenglamalarin tuzing.

b) $y = x^2 + 2x + 12$ funksiyaga $x_0 = 0$ nuqtada o`tkazilgan urinma va normal tenglamalarin tuzing.

c) $y = x^5 + 2x^3 + 12x - 1$ funksiyaga $x_0 = 1$ nuqtada o`tkazilgan urinma va normal tenglamalarin tuzing.

O`z-o`zini tekshirish uchun savollar

1. Funksiya hosilasining ta’rifini ayting.
2. Hosilalar jadvalini yoddan aytib bering.
3. Funksiyalarning differensialining ta’rifini ayting.
4. Hosilaning geometrik va mexanik ma’nosini tushuntiring.
5. Differensialning geometrik ma’nosini tushuntiring.
6. Funksiya grafigining berilgan nuqtadagi urinmasi va normali tenglamalarini yozing.
7. Differensialning taqribiy hisoblashlarga tadbiqlarini tushuntiring.

O`z-o`zini tekshirish uchun testlar

Hosilaga oid testlar

1. $y = f(x)$ funsiyaning x_0 nuqtadagi orttirmasini ko`rsating.

a) $\Delta y = f(x_0 + \Delta x) - f(x_0)$ b) $\Delta y = f(x_0 + \Delta x) + f(x_0)$

c) $\Delta y = f(x_0 + \Delta x)$ d) $\Delta y = f(x_0 - \Delta x) + f(x_0)$

2. $y = f(x)$ funsiyaning x_0 nuqtadagi hosilasini ko`rsating.

a) $y'(x_0) = f'(x_0) = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$.

b) $y'(x_0) = f'(x_0) = \lim\limits_{\Delta x \to 1} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 1} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$.

c) $y'(x_0) = f'(x_0) = \lim\limits_{\Delta x \to 1} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x) + f(x_0)}{\Delta x}$.

d) $y'(x_0) = f'(x_0) = \lim\limits_{\Delta x \to 1} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 1} \frac{f(x_0+\Delta x) + f(x_0)}{\Delta x}$.

3. $\lim\limits_{\Delta x\to 0+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ limit mavjud bo`lsa, bu limit $y=f(x)$ funksiyaning x_0 nuqtadagi … deyiladi va … kabi belgilanadi.

a) o`ng tomonlama hosilasi, $y'_+(x_0)=f'_+(x_0)=\lim\limits_{\Delta x\to 0+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

b) chap tomonlama hosilasi, $y'_+(x_0)=f'_+(x_0)=\lim\limits_{\Delta x\to 0+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

c) o`ng tomonlama hosilasi, $y'_-(x_0)=f'_-(x_0)=\lim\limits_{\Delta x\to 0-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

d) chap tomonlama hosilasi, $y'_-(x_0)=f'_-(x_0)=\lim\limits_{\Delta x\to 0-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

4. $\lim\limits_{\Delta x\to 0-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ limit mavjud bo`lsa, bu limit $y=f(x)$ funksiyaning x_0 nuqtadagi … deyiladi va … kabi belgilanadi.

a) o`ng tomonlama hosilasi, $y'_+(x_0)=f'_+(x_0)=\lim\limits_{\Delta x\to 0+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

b) chap tomonlama hosilasi, $y'_+(x_0)=f'_+(x_0)=\lim\limits_{\Delta x\to 0+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

c) o`ng tomonlama hosilasi, $y'_-(x_0)=f'_-(x_0)=\lim\limits_{\Delta x\to 0-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

d) chap tomonlama hosilasi, $y'_-(x_0)=f'_-(x_0)=\lim\limits_{\Delta x\to 0-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

5. $y=C$ bo`lsa, u holda $y'$ toping. Bunda $C$ – o`zgarmas son.

a) $y'=0$ b) $y'=\alpha x^{\alpha-1}$ c) $y'=\frac{1}{x\ln a}$ d) $y'=\cos x$

6. $y=x^{\alpha}$ bo`lsa, u holda y' toping.

a) $y'=0$ b) $y'=\alpha x^{\alpha-1}$ c) $y'=\frac{1}{x\ln a}$ d) $y'=\cos x$

7. $y=\log_a x$ bo`lsa, u holda y' toping.

a) $y'=0$ b) $y'=\alpha x^{\alpha-1}$ c) $y'=\frac{1}{x\ln a}$ d) $y'=\cos x$

8. $y=a^x$ bo`lsa, u holda y' toping.

a) $y'=a^x\ln a$ b) $y'=\alpha x^{\alpha-1}$ c) $y'=\frac{1}{x\ln a}$ d) $y'=\cos x$

9. $y=\sin x$ bo`lsa, u holda y' toping.

a) $y'=a^x\ln a$ b) $y'=\alpha x^{\alpha-1}$ c) $y'=\frac{1}{x\ln a}$ d) $y'=\cos x$

10. $y=\cos x$ bo`lsa, u holda y' toping.

a) $y'=a^x\ln a$ b) $y'=-\sin x$ c) $y'=\frac{1}{x\ln a}$ d) $y'=\operatorname{tg} x$

11. $y=\operatorname{tg} x$ bo`lsa, u holda y' toping.

a) $y'=\frac{1}{\cos^2 x}$ b) $y'=-\sin x$ c) $y'=-\frac{1}{\sin^2 x}$ d) $y'=\cos x$

12. $y=\operatorname{ctg} x$ bo`lsa, u holda y' toping.

a) $y'=\frac{1}{\cos^2 x}$ b) $y'=-\sin x$ c) $y'=-\frac{1}{\sin^2 x}$ d) $y'=\cos x$

13. $y=\arcsin x$ bo`lsa, u holda y' toping.

a) $y'=\frac{1}{\cos^2 x}$ b) $y'=\frac{1}{\sqrt{1-x^2}}$ c) $y'=-\frac{1}{\sin^2 x}$ d) $y'=\cos x$

14. $y=\arccos x$ bo`lsa, u holda y' toping.

a) $y'=-\frac{1}{\sqrt{1-x^2}}$ b) $y'=\frac{1}{\sqrt{1-x^2}}$ c) $y'=-\frac{1}{\sin^2 x}$ d) $y'=\cos x$

15. $y=\operatorname{arctg} x$ bo`lsa, u holda y' toping.

a) $y' = -\frac{1}{\sqrt{1-x^2}}$ b) $y' = \frac{1}{\sqrt{1-x^2}}$
c) $y' = -\frac{1}{\sin^2 x}$ d) $y' = \frac{1}{1+x^2}$

16. $y = \operatorname{arcctg} x$ bo`lsa, u holda y' toping.

a) $y' = -\frac{1}{\sqrt{1-x^2}}$ b) $y' = -\frac{1}{1+x^2}$
c) $y' = -\frac{1}{\sin^2 x}$ d) $y' = \frac{1}{1+x^2}$

17. $y = f(x)$ funksiyanng x nuqtadagi orttirmasi $\Delta y = A(x)\Delta x + o(\Delta x)\Delta x$ ko`rinishda ifodalash mumkin bo`lsa, u holda $y = f(x)$ funksiya x nuqtada … deyiladi.

a) differensiallanuvchi funksiya b) uzluksiz funksiya
c) cheksiz kichik funksiya d) cheksiz katta funksiya

18. $\Delta y = A(x)\Delta x + o(\Delta x)\Delta x$ tenglikning o`ng tomonidagi asosiy chiziqli $A(x)\Delta x$ hadi $y = f(x)$ funksiyaning … differensiali deyiladi va … orqali belgilanadi.

a) differensiali, $df(x)$ yoki dy b) hosilasi, $df(x)$ yoki dy
c) differensiali, $f'(x)$ yoki y' d) hosilasi, $f'(x)$ yoki y'

19. $y = f(x)$ egri chiziqning $(x_0,\ f(x_0))$ nuqtasidagi urinmasi bilan absissa o`qining musbat yo`nalshi orasidagi burchuk tangensini toping.

a) $tg\alpha = f'(x_0)$ b) $tg\alpha = f(x_0)$ c) $tg\alpha = f'(0)$ d) $tg\alpha = f(0)$

20. $y = f(x)$ funksiyaning x_0 nuqtadagi urinma tenglamasini ko`rsating.

a) $y = f(x_0) + f'(x_0)(x - x_0)$ b) $y = f(x_0) - f'(x_0)(x - x_0)$
c) $y = f(x_0) - f'(x_0)(x + x_0)$ d) $y = f(x_0) + f'(x_0)(x + x_0)$

21. $y = f(x)$ funksiyaning x_0 nuqtadagi normal tenglamasini ko`rsating.

a) $y = f(x_0) - \frac{1}{f'(x_0)}(x - x_0)$ b) $y = f(x_0) + \frac{1}{f'(x_0)}(x - x_0)$
c) $y = f(x_0) + \frac{1}{f'(x_0)}(x + x_0)$ d) $y = f(x_0) - \frac{1}{f'(x_0)}(x + x_0)$

22. $y = x^2 + 3x + 12$ bo`lsa, u holda y' toping.

a) $y' = 2x + 3$ b) $y' = 2x - 3$ c) $y' = 2x$ d) $y' = x + 3$

24. $y = x^5 + \cos 2x$ bo`lsa, u holda y' toping.

a) $y = 5x^4 - 2\sin 2x$ b) $y = 5x^4 + 2\sin 2x$
c) $y = 5x^4 - \sin 2x$ d) $y = 5x^4 - 2\cos 2x$

25. $y = \log_2 x + 4x - 3$ bo`lsa, u holda y' toping.

a) $y' = 0$ b) $y' = \ln 2 + 4$
c) $y' = \frac{1}{x\ln 2} + 4$ d) $y' = \frac{1}{x\ln 2} - 4$

26. $y = 3^x$ bo`lsa, u holda y' toping.

a) $y' = 3^x \ln 3$ b) $y' = 3x^2$
c) $y' = \frac{1}{x\ln 3}$ d) $y' = 3\cos x$

27. $y = \sin 2x + \arcsin 2x$ bo`lsa, u holda y' toping.

a) $y' = 2\cos 2x - \frac{2}{\sqrt{1-4x^2}}$ b) $y' = 2\cos 2x + \frac{1}{\sqrt{1-4x^2}}$
c) $y' = 2\cos 2x + \frac{2}{\sqrt{1-4x^2}}$ d) $y' = \cos 2x - \frac{2}{\sqrt{1-4x^2}}$

28. $y = \operatorname{tg} 3x$ bo`lsa, u holda y' toping.

a) $y' = \frac{3}{\cos^2 x}$ b) $y' = -3\sin x$

c) $y' = -\frac{3}{\sin^2 x}$ d) $y' = 3\cos x$

29. $y = \text{ctg}\, 3x$ bo`lsa, u holda y' toping.

a) $y' = \frac{3}{\cos^2 x}$ b) $y' = -3\sin x$

c) $y' = -\frac{3}{\sin^2 x}$ d) $y' = 3\cos x$

30. $y = \arccos 2x - 3^x$ bo`lsa, u holda y' toping.

a) $y' = -\frac{2}{\sqrt{1-4x^2}} - 3^x \ln 3$ b) $y' = \frac{2}{\sqrt{1-4x^2}} - 3^x \ln 3$

c) $y' = \frac{2}{\sqrt{1-4x^2}} + 3^x \ln 3$ d) $y' = \frac{2}{\sqrt{1-4x^2}} + 3^x \ln 2$

12-§. DIFFERENSIALLASHNING ALGEBRAIK AMALLARGA TA'SIRI

12.1. Yig`ingdi, ko`paytma va bo`linmaning differensiallanishi (nazariy qism)

O`zining aniqlanish sohasidagi barcha nuqtalarida differensiallanuvchi bo`lgan funksiya *differensiallanuvchi funksiya* deyiladi.

Hosilani ta'rifdan foydalanib topish odatda sodda tuzilishga ega funksiyalar uchun qulay bo`ladi. Shuning uchun hosilalarni osonroq topish qoidalarini keltiramiz.

1. Agar k biror o`zgarmas, $y = f(x)$ differensiallanuvchi funksiya bo`lsa, u holda $y = kf(x)$ funksiya ham differensiallanuvchi va ushbu tenglik o`rinli bo`ladi

$$\boldsymbol{\big(kf(x)\big)' = kf'(x).}$$

2. Agar $u(x)$ va $v(x)$ differensiallanuvchi funksiyalar bo`lsa, $u(x) \pm v(x)$ **algebraik yig`indi** ham differensiallanuvchi funksiya bo`ladi, vaholanki

$$\boldsymbol{\big(u(x) \pm v(x)\big)' = u'(x) \pm v'(x).}$$

3. Agar $u(x)$ va $v(x)$ differensiallanuvchi funksiyalar bo`lsa, $u(x) \cdot v(x)$ **ko`paytma** ham differensiallanuvchi funksiya bo`ladi, bunda

$$\boldsymbol{\big(u(x)v(x)\big)' = u'(x)v(x) + u(x)v'(x).}$$

4. Agar $u(x)$ va $v(x)$ differensiallanuvchi funksiyalar bo`lib, $v(x) \neq 0$ bo`lsa, $\boldsymbol{\frac{u(x)}{v(x)}}$ **bo`linma** ham differensiallanuvchi funksiya bo`ladi, bu holda

$$\boldsymbol{\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}.}$$

Endi differensialni topish qoidalarini keltiramiz. Bu qoidalar hosila belgisi $'$ o`rniga differensial belgisi $d$ ni qo`yilsa, hosilani topish qoidalariga o`xshash bo`ladi.

1. $d(ku(x)) = kdu(x)$, bunda k – o`zgarmas funksiya.
2. $d\big(u(x) \pm v(x)\big) = du(x) \pm dv(x)$.
3. $d\big(u(x)v(x)\big) = v(x)du(x) + u(x)dv(x)$.
4. $d\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)du(x) - u(x)dv(x)}{v^2(x)}, \qquad v(x) \neq 0.$

12.2. Oshkormas berilgan funksiyani differensiallash

Aytaylik, z o`zgaruvchi bir-biriga bog`liqmas x, y, ... o`zgaruvchi miqdorlarga bog`liq bo`lsin. $z$ o`zgaruvchini quyidagicha bitta o`zgaruvchi bo`yicha differensiallash mumkin: faqatgina x ning o`zgarishini e'tiborga olib, $x$ bo`yicha, faqatgina y ning o`zgarishini e'tiborga olib, $y$ bo`yicha, va hakozo. Natijada bitta z funksiyaning bir nechta turlicha hosilalari paydo bo`ladi. Bu hosilalarni bir-biridan farqlash va muayyan hosila qaysi o`zgaruvchi bo`yicha olinganligini ko`rsatib turishi uchun z'_x, z'_y, ... belgilashlar ishlatiladi. z'_x yozuv z funksiyaning x bo`yicha hosilasini anglatadi, $z'_y$ yozuv $z$ funksiyaning $y$ bo`yicha hosilasini bildiradi va hakozo.

$y = y(x)$ funksiya oshkormas, ya'ni y ga nisbatan yechilmagan $F(x, y) = 0$ tenglama bilan berilgan bo`lsin. $y = y(x)$ differensiallanuvchi funksiya bo`lib, uning

y'_x hosilasini topish talab etilsin. Buning uchun avvalo $F(x,\ y)$ ifodani ikkita x va y o`zgaruvchilarga bog`liq funksiya deb hisoblab, $F'_x(x,y)$ va $F'_y(x,y)$ hosilalar topiladi. Keyin talab etilayotgan y'_x hosila

$$\boxed{\boldsymbol{y'_x = -\frac{F'_x(x,y)}{F'_y(x,y)}}}$$

formuladan topiladi. Odatda y'_x yozuv o`rniga $\frac{dy}{dx}$ belgi ham ishlatiladi: $y'_x = \frac{dy}{dx}$.

12.2.1.-misol. $y = 2x^3 + 3x^5 + x$ bo`lsa, x'_y hosilani toping.

Yechish. $y = 2x^3 + 3x^5 + x$ tenglama $x = x(y)$ ga nisbatan yechilmagan. Uni $F(x,y) = 0$ ko`rinishda yozamiz. Bu yerda $F(x,y) = 2x^3 + 3x^5 + x - y$. Endi $F'_x(x,y)$ va $F'_y(x,y)$ hosilalar topamiz:

$$F'_x(x,y) = 6x^2 + 15x^4 + 1, \qquad F'_y(x,y) = -1.$$

So`ralayotgan x'_y hosilani $x'_y = -\frac{F'_y(x,y)}{F'_x(x,y)}$ tenglikdan topamiz:

$$x'_y = -\frac{F'_y(x,y)}{F'_x(x,y)} = -\frac{-1}{6x^2+15x^4+1} = \frac{1}{6x^2+15x^4+1}.$$

12.2.2.-Misol. $\sin xy + \cos xy = 0$ tenglama bilan oshkormas berilgan $y = y(x)$ funksiyaning y'_x hosilasini toping.

Yechish. $F(x,y) = \sin xy + \cos xy$ deb,

$$F'_x(x,y) = y\cos xy - y\sin xy = y(\cos xy - \sin xy),$$
$$F'_y(x,y) = x\cos xy - x\sin xy = x(\cos xy - \sin xy)$$

hosilalarni topamiz va $y'_x = -\frac{F'_x(x,y)}{F'_y(x,y)}$ formuladan izlangan hosilani topamiz:

$$y'_x = -\frac{y(\cos xy - \sin xy)}{x(\cos xy - \sin xy)} = -\frac{y}{x}.$$

12.3. Murakkab funksiyaning differensiallanishi

y ning x dan murakkab funksiyasini qaraymiz: $y = f(u)$, $u = g(x)$.

Agar $u = g(x)$ funksiya x_0 nuqtada $g'_x(x_0)$ hosilaga, $y = f(u)$ funksiya esa $u_0 = g(x_0)$ nuqtada $f'_u(u_0)$ hosilaga ega bo`lsa, u holda $y = f\big(g(x)\big)$ murakkab funksiya x_0 nuqtada

$$f'_x(g(x_0)) = g'_x(x_0) \cdot f'_u(u_0)$$

hosilaga ega.

Demak, **differensiallanuvchi $\boldsymbol{y = f(u)}$ va $\boldsymbol{u = g(x)}$ funksiyalarning murakkab funksiyasi differensiallanuvchi bo`ladi:**

$$\boxed{\boldsymbol{f'_x(g(x)) = g'_x(x) \cdot f'_u(u)}},$$

yoki qisqalik uchun

$$\boxed{\boldsymbol{y'_x = u'_x \cdot y'_u}}.$$

12.3.1.-misol. $y = \sin\ln x$ funksiya hosilasini toping.

Yechish. Bu funksiyani oraliq u o`zgaruvchi kiritib, $y = \sin u$, $u = \ln x$ singari yozish mumkin. U holda $u'_x = \frac{du}{dx} = \frac{1}{x}$ va $y'_u = \frac{dy}{du} = \cos u$ bo`lib, formuladan ushbuga egamiz:

$$y'_x = \frac{dy}{dx} = \frac{1}{x} \cdot \cos u = \frac{1}{x} \cos \ln x.$$

Quyidagi formulalarda $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ deb hisoblaymiz.

1. $(u^\alpha)' = \alpha u' u^{\alpha-1}$; xususan, $\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$, $(\sqrt{u})' = \frac{u'}{2\sqrt{u}}$.
2. $(\log_a u)' = \frac{u'}{u \ln a}$; xususan, $(\ln u)' = \frac{u'}{u}$.
3. $(a^u)' = a^u u' \ln a$; xususan, $(\mathrm{e}^u)' = u' \mathrm{e}^u$.
4. $(\sin u)' = u' \cos u$.
5. $(\cos u)' = -u' \sin u$.
6. $(\mathrm{tg}\, u)' = \frac{u'}{\cos^2 u}$.
7. $(\mathrm{ctg}\, u)' = -\frac{u'}{\sin^2 u}$.
8. $(\arcsin u)' = \frac{u'}{\sqrt{1-u^2}}$.
9. $(\arccos u)' = -\frac{u'}{\sqrt{1-u^2}}$.
10. $(\mathrm{arctg}\, u)' = \frac{u'}{1+u^2}$.
11. $(\mathrm{arcctg}\, u)' = -\frac{u'}{1+u^2}$.

12.4. Teskari funksiyaning differensiallanishi

Agar $y = f(x)$ funksiya biror oraliqda uzluksiz, qat'iy monoton va biror x nuqtada $f'_x(x) \neq 0$ hosilaga ega bo`lsa, u holda uning teskari $x = g(y)$ funksiyasi ham $y$ nuqtada $g'_y(y)$ hosilaga ega hamda ushbu formula o`rinli bo`ladi

$$\boxed{g'_y(y) = \frac{1}{f'_x(x)}}.$$

12.4.1.-misol. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ kesmada qaralayotgan $y = \sin x$ funksiyaning hosilasi $y' = \cos x$ ekanligini bilgan holda unga teskari bo`lgan funksiya hosilasini toping.

Yechish. $y = \sin x$ funksiyaning $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ kesmadagi teskari funksiyasi $x = \arcsin y$ bo`ladi. Teskari funksiya hosilasi formulasidan:

$$x'_y = (\arcsin y)'_y = \frac{1}{(\sin x)'_x} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}},$$

ya'ni $(\arcsin y)'_y = \frac{1}{\sqrt{1-y^2}}$. Endi y ni x bilan almashtirb,

$$\arcsin' x = \frac{1}{\sqrt{1 - x^2}}.$$

tenglikka ega bo`lamiz.

12.5. Logarifmik differensiallash

Ayrim hollarda funksiyani differensiallashdan oldin, uni logarifmlash qulay bo`ladi. $y = f(x)$ differensiallanuvhi funksiya bo`lib, uning hosilasini topish talab

etilsin. Avvalo uni logarifmlaymiz: $\ln y = \ln f(x)$. Bu tenglikning har bir qismini differensiallaymiz. Bu yerda $\ln y$ ifodaning hosilasi murakkab funksiya sifatida topiladi: $\frac{y'}{y} = (\ln f(x))'$. Bu tenglikdan $y' = y(\ln f(x))'$ yoki

$$\boxed{\boldsymbol{y' = f(x)(\ln f(x))'}}.$$

Differensiallashning bu usuli *logarifmik differensiallash* deyiladi.

Ba'zan logarifmik differensiallashni qo`llamasdan differensiallash qiyinchilik tug`diradi. Xususan:

Darajali-ko`rsatkichli deb ataluvchi $y = \left(u(x)\right)^{v(x)}$ ko`rinishdagi funksiyalarni differensiallashda shunday bo`ladi, bunda $u(x)$ va $v(x)$ bitta $x$ o`zgaruvchiga bog`liq qandaydir differensiallnuvchi funksiyalar. Bu holda:

$$\ln y = \ln\left(u(x)\right)^{v(x)} = v(x)\ln u(x),$$

$$\frac{y'}{y} = v'(x)\ln u(x) + v(x)\cdot\frac{u'(x)}{u(x)},$$

$$y' = y\left(v'(x)\ln u(x) + v(x)\cdot\frac{u'(x)}{u(x)}\right),$$

$$\boxed{y' = u(x)^{v(x)}\left(v'(x)\ln u(x) + v(x)\cdot\frac{u'(x)}{u(x)}\right)};$$

Yoki funksiya juda ko`p ko`paytuvchiga ega bo`lsa, ya'ni

$$y = f_1(x)f_2(x)\cdot\ldots\cdot f_k(x)$$

ko`rinishga ega bo`lsa. Bu holda:

$$\ln y = \ln\left(f_1(x)f_2(x)\cdot\ldots\cdot f_k(x)\right) = \ln f_1(x) + \ln f_2(x) + \cdots + \ln f_k(x),$$

$$\frac{y'}{y} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \cdots + \frac{f_k'(x)}{f_k(x)},$$

$$y' = y\left(\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \cdots + \frac{f_k'(x)}{f_k(x)}\right),$$

$$\boxed{y' = f_1(x)f_2(x)\cdot\ldots\cdot f_k(x)\cdot\left(\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \cdots + \frac{f_k'(x)}{f_k(x)}\right)}.$$

12.5.1.-misol. Quyidagi funksiyalarni differensiallang:

$$a)\ y = (tg\ x)^{ctgx};$$

$$b)\ y = \sqrt[5]{\left(\frac{x+1}{x-1}\right)^2}\left(\frac{x-2}{x+2}\right)^3\sqrt[4]{\left(\frac{1-\sin x}{1-\cos x}\right)^3}.$$

Yechish. $a)$ $y = (\mathrm{tg}\,x)^{\mathrm{ctg}\,x}$ tenglikning ikkala tomonini logarifmlab, $\ln y = \mathrm{ctg}\,x\ln\mathrm{tg}\,x$ tenlikka ega bo`lamiz. Uning ikkala qismidan hosila olamiz:

$$\frac{y'}{y} = -\frac{1}{\sin^2 x}\cdot\ln\mathrm{tg}\,x + \mathrm{ctg}\,x\cdot\frac{\frac{1}{\cos^2 x}}{\mathrm{tg}\,x} = -\frac{\ln\mathrm{tg}\,x}{\sin^2 x} + \frac{1}{\sin^2 x} = \frac{1-\ln\mathrm{tg}\,x}{\sin^2 x}.$$

Bu yerdan

$$y' = (\mathrm{tg}\,x)^{\mathrm{ctg}x}\cdot\frac{1-\ln\mathrm{tg}\,x}{\sin^2 x}.$$

$b)$ Berilgan funksiyani logarifmlab,

$$\ln y = \ln\left(\sqrt[5]{\left(\frac{x+1}{x-1}\right)^2}\left(\frac{x-2}{x+2}\right)^3\sqrt[4]{\left(\frac{1-\sin x}{1-\cos x}\right)^3}\right) = \ln\left(\left(\frac{x+1}{x-1}\right)^{\frac{2}{5}}\cdot\left(\frac{x-2}{x+2}\right)^3\cdot\left(\frac{1-\sin x}{1-\cos x}\right)^{\frac{3}{4}}\right) =$$

$$= \frac{2}{5}\ln\frac{x+1}{x-1} + 3\ln\frac{x-2}{x+2} + \frac{3}{4}\ln\frac{1-\sin x}{1-\cos x},$$

ya'ni $\ln y = \frac{2}{5}\ln\frac{x+1}{x-1} + 3\ln\frac{x-2}{x+2} + \frac{3}{4}\ln\frac{1-\sin x}{1-\cos x}$ tenglikni hosil qilamiz. Uning ikkala tarafini differensiallab, so`raglan hosilani topamiz:

$$\frac{y'}{y} = \frac{2}{5}\cdot\frac{\frac{x-1-(x+1)}{(x-1)^2}}{\frac{x+1}{x-1}} + 3\cdot\frac{\frac{x+2-(x-2)}{(x+2)^2}}{\frac{x-2}{x+2}} + \frac{3}{4}\cdot\frac{\frac{-\cos x(1-\cos x)-\sin x(1-\sin x)}{(1-\cos x)^2}}{\frac{1-\sin x}{1-\cos x}},$$

$$y' = y\left(\frac{-4}{5(x+1)(x-1)} + \frac{12}{(x-2)(x+2)} + \frac{3(-\cos x+\cos^2 x-\sin x+\sin^2 x)}{4(1-\sin x)(1-\cos x)}\right) =$$

$$= \sqrt[5]{\left(\frac{x+1}{x-1}\right)^2}\left(\frac{x-2}{x+2}\right)^3\sqrt[4]{\left(\frac{1-\sin x}{1-\cos x}\right)^3}\left(\frac{12}{x^2-4} - \frac{4}{5(x^2-1)} + \frac{3(1-\cos x-\sin x)}{4(1-\sin x)(1-\cos x)}\right),$$

ya'ni $y' = \sqrt[5]{\left(\frac{x+1}{x-1}\right)^2}\left(\frac{x-2}{x+2}\right)^3\sqrt[4]{\left(\frac{1-\sin x}{1-\cos x}\right)^3}\left(\frac{12}{x^2-4} - \frac{4}{5(x^2-1)} + \frac{3(1-\cos x-\sin x)}{4(1-\sin x)(1-\cos x)}\right)$.

12.6. Parametrik ko`rinishda berilgan funksiyani differensiallash

$\varphi(t)$ va $\psi(t)$ lar $(\alpha,\ \beta)$ intervalda differensiallanuvchi funksiyalar bo`lib, $y = y(x)$ funksiya ushbu

$$x = \varphi(t),\ y = \psi(t)$$

parametrik ko`rinishda berilgan hamda barcha $t \in (\alpha,\ \beta)$ larda $\varphi'(t) \neq 0$ bo`lsa, u holda y'_x hosila mavjud hamda bu hosila

$$\boxed{y'_x = \frac{y'_t}{x'_t}} \text{ yoki } \boxed{y'_x = \frac{\psi'(t)}{\varphi'(t)}}$$

formuladan topiladi.

12.6.1.-Misol. Quyidagi tenglamalar $y = y(x)$ funksiyani aniqlaydi. y funksiyaning x bo`yicha hosilasi y'_x ni toping:

$a)\ x = a(t - \sin t),\qquad y = a(1 - \cos t);$

$b)\ x = k\sin t - \sin kt,\qquad y = k\cos t + \cos kt;$

$c)\ x = 2\ln \mathrm{ctg} t,\qquad y = \mathrm{tg}\, t + \mathrm{ctg}\, t;$

$d)\ x = \mathrm{e}^{ct},\qquad y = \mathrm{e}^{-ct}.$

Bu misollarda a, c, k – o`zgarmaslar, t – parametr.

Yechish. $a)$ x va y lardan t bo`yicha hosilalarni topamiz:

$$x'_t = a(1 - \cos t),\qquad y'_t = a\sin t.$$

Topilgan bu hosilalarni formulaga qo`yib, talab etilgan hosilani topamiz:

$$y'_x = \frac{y'_t}{x'_t} = \frac{a\sin t}{a(1-\cos t)},\qquad \text{bu yerda } t \neq 2k\pi,\qquad k = 0,\ \pm1,\ \pm2, \dots .$$

$$b)\ \frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{(k\cos t+\cos kt)'_t}{(k\sin t-\sin kt)'_t} = -\frac{k\sin t+k\sin kt}{k\cos t-k\cos kt} = -\frac{\sin t+\sin kt}{\cos t-\cos kt},$$

bu yerda $(k-1)t \neq \pi n,\qquad n = 0,\ \pm1,\ \pm2, \dots .$

$$c)\ y'_x = \frac{y'_t}{x'_t} = \frac{(\mathrm{tg}\, t+\mathrm{ctg}\, t)'_t}{(2\ln \mathrm{ctg}\, t)'_t} = \frac{\frac{1}{\cos^2 t}-\frac{1}{\sin^2 t}}{2\cdot\frac{-\frac{1}{\sin^2 t}}{\mathrm{ctg}\, t}} = \frac{\sin^2 t-\cos^2 t}{\cos^2 t\sin^2 t}\cdot\frac{\sin t\cos t}{-2} = \frac{\cos 2t}{\sin 2t} = \mathrm{ctg}\, 2t,$$

bu yerda $t \neq \frac{\pi k}{2},\qquad k = 0,\ \pm1,\ \pm2, \dots .$

$$d)\ \frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{(\mathrm{e}^{-ct})'_t}{(\mathrm{e}^{ct})'_t} = \frac{-c\mathrm{e}^{-ct}}{c\mathrm{e}^{ct}} = -\mathrm{e}^{-2ct},\qquad \text{bu yerda } t \in (-\infty, +\infty).$$

12.7. Differensial formaning invariantligi

Aytaylik, $y = f(u)$ va $u = g(x)$ funksiyalar $y = y(x) = f\big(g(x)\big)$ murakkab funksiyani hosil qilsin. Undan tashqari, $f'(u) = f'_u(u)$ va $g'(x) = g'_x(x)$ hosilalar mavjud bo`lsin. U holda $f\big(g(x)\big)$ murakkab funksiya differensiallanuvchi bo`ladi va uning x bo`yicha hosilasi

$$f'_x\big(g(x)\big) = f'_u(u) \cdot g'_x(x) = f'_{g(x)}\big(g(x)\big) \cdot g'_x(x)$$

formula bilan topiladi.

Endi murakkab funksiya differensialini topish bilan shug`ullanamiz. Bu ishni ikkita usulda qilish mumkin:

(i) y ning x bo`yicha differensialini topamiz: $dy = df\big(g(x)\big) = f'_x(g(x))dx$. Bu holda $f'_x(g(x)) = f'_{g(x)}\big(g(x)\big) \cdot g'_x(x)$ formulani qo`llab,

$$df\big(g(x)\big) = f'_{g(x)}\big(g(x)\big) \cdot g'_x(x)dx$$

tenglikka erishamiz;

(ii) avval y ning u bo`yicha $dy = df(u) = f'_u(u)du$ differensialini, keyin $u$ ning $x$ bo`yicha $du = dg(x) = g'_x(x)dx$ differensialini topamiz. $du = g'_x(x)dx$ differensialni $df(u) = f'_u(u)du$ ga qo`yib, va $u = g(x)$ ekanligini e'tiborga olib,

$$df\big(g(x)\big) = f'_{g(x)}\big(g(x)\big) \cdot g'_x(x)dx$$

tenglikni hosil qilamiz.

Ikkala usulda ham bir xil tenglik kelib chiqdi. Demak, $y = f(u)$ funksiya uchun u o`zgaruvchi hech qanday o`zgaruvchiga bog`liq bo`lmagan **erkli o`zgaruvchi** yoki boshqa biror o`zgaruvchiga **bog`liq o`zgaruvchi** bo`lishidan qat'iy nazar, $y$ funksiyaning differensialini $dy = f'(u)du$ shaklda yozish mumkin ekan. Bu xossa *differensial formasining invariantligi* deyiladi. Ikkala holning bir-biridan farqi shundaki, $u$ erkli o`zgaruvchi bo`lsa, $du = \Delta u$ ($u$ ning differensiali $u$ ning orttirmasiga teng), $u$ boshqa biror $x$ o`zgaruvchiga bog`liq $u = g(x)$ funksiya bo`lsa, du differensial $g(x)$ funksiyaning differensialiga teng bo`ladi.

12.7.1.-misol. u va v lar x ning differensiallanuvchi funksiyalari bo`lsin. $y = \operatorname{arctg}\frac{u}{v}$ funksiyaning differensialini toping.

Differensialning invariantligi xossasidan foydalanib, ushbuni topamiz

$$dy = d\left(\operatorname{arctg}\frac{u}{v}\right) = \left(\operatorname{arctg}\frac{u}{v}\right)'_{\frac{u}{v}} d\left(\frac{u}{v}\right) = \frac{1}{1+\left(\frac{u}{v}\right)^2} \cdot \frac{vdu - udv}{v^2} = \frac{vdu - udv}{u^2+v^2}.$$

Bu tenglikka $u = u(x)$, $v = v(x)$, $du = u'(x)dx$, $dv = v'(x)dx$ larni qo`yib,

$$dy = \frac{v(x)u'(x) - u(x)v'(x)}{u^2(x)+v^2(x)}dx$$

differensialni topamiz.

Mustahkamlash uchun mashqlar

Hosilaning amallarga ta'siriga oid masalalar

1. Funksiya hosilasini toping.

1. $y = x^3 \sin 3x$ 2. $y = e^x tg\, 4x$ 3. $y = \sqrt[2]{x^4 sin^4 x}$
4. $y = xctg^2 7x$ 5. $y = -2^{cos^4 5x}$ 6. $y = e^{arctg\sqrt{x}} 10$
7. $y = (2^{x^4} - tg^4 x)^3$ 8. $y = ln^5(x - 2^{-x})$ 9. $y = \sin(tg\sqrt{x})$
10. $y = xsin^2 x \cdot 2^{x^2}$ 11. $y = 2^{x/lnx}$ 12. $y = arctg\sqrt{1 + x^2}$
13. $y = e^{-\sqrt{x^2+2x+2}}$ 14. $y = sh^3 x^2$ 15. $y = (2^{tg3x} + tg3x)^2$

2. Logarifmik differensiallash usulidan foydalanib funksiyalarni differensiallang.

1. $y = (\sin 3x)^{\cos 5x}$ 2. $y = (x^3 + 1)^{tg\, 2x}$
3. $y = (\cos(x + 2))^{\ln x}$ 4. $y = (\sin 3x)^{\arccos x}$
5. $y = (\cos 5x)^{arctgx}$ 6. $y = \left(\sqrt{3x + 2}\right)^{arctg3x}$
7. $= \left(\ln\sqrt{3x + 2}\right)^{ctg3x}$ 8. $y = (\ln(x + 4))^{ctg7x}$

3. Oshkormas ko`rinishida berilgan funksiyani differensiallang.

1. $e^{xy} - x^3 - y^3 = 3;$
2. $xy - arctg\frac{x}{y} = 3;$
3. $\sqrt[3]{x} + \sqrt[3]{y} = a;$
4. $\sqrt[3]{x + 1} + \sqrt[3]{y - 5} = xy.$

4. Parametrik ko`rinishda berilgan funksiyani differensiallang

1. $\begin{cases} y = t^3 + t^2 - 1 \\ y = t^2 + t + 1 \end{cases}$
2. $\begin{cases} y = 2sin^3 t \\ x = 2cos^3 t \end{cases}$
3. $\begin{cases} y = (2t + 3)sin^2 t \\ x = 3t^2 \end{cases}$
4. $\begin{cases} y = 6sin^3 t \\ x = 2cos^3 t + 3 \end{cases}$

O`z-o`zini tekshirish uchun savollar

1. Yig`ingdi, ko`paytma va bo`linmaning differensiallanishiga oid formulalarini yozing.
2. Oshkormas berilgan funksiyani differensiallash formulasini yozing.
3. Murakkab funksiyaning differensiallanishiga oid formulalarini yozing.
4. Teskari funksiyaning differensiallanishiga oid formulalarini yozing.
5. Logarifmik differensiallashni tushuntiring.
6. Parametrik ko`rinishda berilgan funksiyani differensiallash formulalarini yozing.

O`z-o`zini tekshirish uchun testlar

Hosilaning amallarga ta'siriga oid testlar

1. Agar k biror o`zgarmas, $y = f(x)$ differensiallanuvchi funksiya bo`lsa, u holda $y = kf(x)$ funksiya ham differensiallanuvchi va ... tenglik o`rinli bo`ladi

a)$\left(kf(x)\right)' = kf'(x)$ b)$\left(kf(x)\right)' = f'(kx)$

c)$\left(kf(x)\right)' = -kf'(x)$ d)$\left(kf(x)\right)' = -f'(kx)$

2. Agar $u(x)$ va $v(x)$ differensiallanuvchi funksiyalar bo`lsa, $u(x) \pm v(x)$ algebraik yig`indi ham differensiallanuvchi funksiya va ... tenglik o`rinli bo`ladi.

a) $\left(u(x) \pm v(x)\right)' = u'(x) \pm v'(x)$

b) $\left(u(x) \pm v(x)\right)' = ku'(x) \pm kv'(x)$

c) $\left(u(x) \pm v(x)\right)' = u'(x)v(x) + u(x)v'(x)$

d) $\left(u(x) \pm v(x)\right)' = u'(x)v(x) - u(x)v'(x)$

3. Agar $u(x)$ va $v(x)$ differensiallanuvchi funksiyalar bo`lsa, $u(x) \cdot v(x)$ ko`paytma ham differensiallanuvchi funksiya va ... tenglik o`rinli bo`ladi.

a) $\left(u(x)v(x)\right)' = u'(x)v(x) + u(x)v'(x)$

b) $\left(u(x)v(x)\right)' = u'(x)v(x) - u(x)v'(x)$

c) $\left(u(x)v(x)\right)' = \frac{u'(x)v(x)-u(x)v'(x)}{v^2(x)}$

d) $\left(u(x)v(x)\right)' = u'(x) \pm v'(x)$

4. Agar $u(x)$ va $v(x)$ differensiallanuvchi funksiyalar bo`lib, $v(x) \neq 0$ bo`lsa, $\frac{u(x)}{v(x)}$ bo`linma ham differensiallanuvchi funksiya va ... tenglik o`rinli bo`ladi.

a) $\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x)-u(x)v'(x)}{v^2(x)}$

b) $\left(\frac{u(x)}{v(x)}\right)' = u'(x)v(x) + u(x)v'(x)$

c) $\left(\frac{u(x)}{v(x)}\right)' = u'(x)v(x) - u(x)v'(x)$

d) $\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x)+u(x)v'(x)}{v^2(x)}$

5. $y = y(x)$ funksiya oshkormas, ya'ni y ga nisbatan yechilmagan $F(x,\ y) = 0$ tenglama bilan berilgan bo`lsa, u holda y'_x hosilani toping.

a) $y'_x = -\frac{F'_x(x,y)}{F'_y(x,y)}$ b) $y'_x = \frac{F'_x(x,y)}{F'_y(x,y)}$

c) $y'_x = \frac{F'_x(x,y)}{2F'_y(x,y)}$ d) $y'_x = -\frac{2F'_x(x,y)}{F'_y(x,y)}$

6. Agar $u = g(x)$ funksiya x_0 nuqtada $g'_x(x_0)$ hosilaga, $y = f(u)$ funksiya esa $u_0 = g(x_0)$ nuqtada $f'_u(u_0)$ hosilaga ega bo`lsa, u holda $y = f\left(g(x)\right)$ murakkab funksiya x_0 nuqtadagi hosilasini topng.

a) $f'_x(g(x_0)) = g'_x(x_0) \cdot f'_u(u_0)$ b) $f'_x\left(g(x_0)\right) = g'_x(x_0) - f'_u(u_0)$

c) $f'_x\left(g(x_0)\right) = g'_x(x_0) + f'_u(u_0)$ d) $f'_x\left(g(x_0)\right) = g'_x(x_0) : f'_u(u_0)$

7. Agar $y = f(x)$ funksiya biror oraliqda uzluksiz, qat'iy monoton va biror x nuqtada $f'_x(x) \neq 0$ hosilaga ega bo`lsa, u holda uning teskari $x = g(y)$ funksiyasi ham $y$ nuqtada $g'_y(y)$ hosilaga ega hamda … formula o`rinli bo`ladi.

a) $g'_y(y) = \frac{1}{f'_x(x)}$ b) $g'_y(y) = -\frac{1}{f'_x(x)}$

c) $g'_y(y) = \frac{2}{f'_x(x)}$ d) $g'_y(y) = -\frac{2}{f'_x(x)}$

8. $y = f(x)$ differensiallanuvhi funksiya uchun logarifmik differensiallash formulasini ko`rsating.

a) $y' = f(x)(\ln f(x))'$ b) $y' = -f(x)(\ln f(x))'$

c) $y' = -f(x)(-\ln f(x))'$ d) $y' = f(x)(-\ln f(x))'$

9. $y = \left(u(x)\right)^{v(x)}$ ko`rinishdagi funksiyalarni differensiallash uchun logarifmik differensiallash formulasini ko`rsating.

a) $y' = u(x)^{v(x)}\left(v'(x)\ln u(x) + v(x)\cdot\frac{u'(x)}{u(x)}\right)$

b) $y' = u(x)^{v(x)}\left(v'(x)\ln u(x) - v(x)\cdot\frac{u'(x)}{u(x)}\right)$

c) $y' = -u(x)^{v(x)}\left(v'(x)\ln u(x) + v(x)\cdot\frac{u'(x)}{u(x)}\right)$

d) $y' = -u(x)^{v(x)}\left(v'(x)\ln u(x) - v(x)\cdot\frac{u'(x)}{u(x)}\right)$

10. $y = f_1(x)f_2(x)\cdot\ldots\cdot f_k(x)$ ko`rinishga ega funksiyalarni differensiallash uchun logarifmik differensiallash formulasini ko`rsating.

a) $y' = f_1(x)f_2(x)\cdot\ldots\cdot f_k(x)\cdot\left(\frac{f'_1(x)}{f_1(x)} + \frac{f'_2(x)}{f_2(x)} + \cdots + \frac{f'_k(x)}{f_k(x)}\right)$

b) $y' = f_1(x)f_2(x)\cdot\ldots\cdot f_k(x)\cdot\left(\frac{f'_1(x)}{f_1(x)} - \frac{f'_2(x)}{f_2(x)} - \cdots - \frac{f'_k(x)}{f_k(x)}\right)$

c) $y' = -f_1(x)f_2(x)\cdot\ldots\cdot f_k(x)\cdot\left(\frac{f'_1(x)}{f_1(x)} + \frac{f'_2(x)}{f_2(x)} + \cdots + \frac{f'_k(x)}{f_k(x)}\right)$

d) $y' = -f_1(x)f_2(x)\cdot\ldots\cdot f_k(x)\cdot\left(\frac{f'_1(x)}{f_1(x)} - \frac{f'_2(x)}{f_2(x)} - \cdots - \frac{f'_k(x)}{f_k(x)}\right)$

11. $\varphi(t)$ va $\psi(t)$ lar $(\alpha,\ \beta)$ intervalda differensiallanuvchi funksiyalar bo`lib, $y = y(x)$ funksiya ushbu $x = \varphi(t)$, $y = \psi(t)$ parametrik ko`rinishda berilgan hamda barcha $t \in (\alpha,\ \beta)$ larda $\varphi'(t) \neq 0$ bo`lsa, u holda y'_x hosilani toping.

a) $y'_x = \frac{\psi'(t)}{\varphi'(t)}$ b) $y'_x = -\frac{\psi'(t)}{\varphi'(t)}$

c) $y'_x = -\frac{\psi\ \ (t)}{\varphi'(t)}$ d) $y'_x = -\frac{\psi'(t)}{\varphi\ \ (t)}$

12. $y = u^\alpha$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = \alpha u' u^{\alpha-1}$ b) $y' = \frac{u'}{u\ln a}$

c) $y' = a^u u' \ln a$ d) $y' = u'\cos u$

13. $y = \log_a u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = \alpha u' u^{\alpha-1}$ b) $y' = \frac{u'}{u\ln a}$

c) $y' = a^u u' \ln a$ d) $y' = u'\cos u$

14. $y = \ln u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = \frac{u'}{u}$ b) $y' = \frac{u'}{u \ln a}$ c) $y' = a^u u' \ln a$ d) $y' = u' \cos u$

15. $y = a^u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = \frac{u'}{u}$ b) $y' = \frac{u'}{u \ln a}$ c) $y' = a^u u' \ln a$ d) $y' = u' \cos u$

16. $y = \mathrm{e}^u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = u'\mathrm{e}^u$ b) $y' = \frac{u'}{u \ln a}$ c) $y' = a^u u' \ln a$ d) $y' = u' \cos u$

17. $y = \sin u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = u'\mathrm{e}^u$ b) $y' = \frac{u'}{u \ln a}$ c) $y' = a^u u' \ln a$ d) $y' = u' \cos u$

18. $y = \cos u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = u'\mathrm{e}^u$ b) $y' = -u' \sin u$ c) $y' = a^u u' \ln a$ d) $y' = u' \cos u$

19. $y = \mathrm{tg}\, u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = \frac{u'}{\cos^2 u}$ b) $y' = -u' \sin u$ c) $y' = -\frac{u'}{\sin^2 u}$ d) $y' = u' \cos u$

20. $y = \mathrm{ctg}\, u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = \frac{u'}{\cos^2 u}$ b) $y' = -u' \sin u$ c) $y' = -\frac{u'}{\sin^2 u}$ d) $y' = u' \cos u$

21. $y = \arcsin u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = \frac{u'}{\cos^2 u}$ b) $y' = -u' \sin u$

c) $y' = -\frac{u'}{\sin^2 u}$ d) $y' = \frac{u'}{\sqrt{1-u^2}}$

22. $y = \arccos u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = -\frac{u'}{\sqrt{1-u^2}}$ b) $y' = -u' \sin u$

c) $y' = -\frac{u'}{\sin^2 u}$ d) $y' = \frac{u'}{\sqrt{1-u^2}}$

23. $y = \mathrm{arctg}\, u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = -\frac{u'}{\sqrt{1-u^2}}$ b) $y' = -\frac{u'}{1+u^2}$

c) $y' = \frac{u'}{1+u^2}$ d) $y' = \frac{u'}{\sqrt{1-u^2}}$

24. $y = \mathrm{arcctg}\, u$ da $u = u(x)$ va $u' = u'_x = \frac{du}{dx}$ bo`lsa, u holda y' toping.

a) $y' = -\frac{u'}{\sqrt{1-u^2}}$ b) $y' = -\frac{u'}{1+u^2}$

c) $y' = \frac{u'}{1+u^2}$ d) $y' = \frac{u'}{\sqrt{1-u^2}}$

25. $y = (2x+1)^5 + \cos(2x+1)$ bo`lsa, u holda y' toping.

a) $y = 10(2x+1)^4 - 2\sin(2x+1)$

b) $y = 10(2x+1)^4 + 2\sin(2x+1)$

c) $y = 10(2x+1)^4 + 2\sin(2x-1)$

d) $y = -10(2x+1)^4 + 2\sin(2x+1)$

26. $y = \ln(\sin 2x) + 4x - 3$ bo`lsa, u holda y' toping.

a) $y' = 2\text{ctg}\, 2x + 4$ b) $y' = \ln cos 2x + 4$

c) $y' = \frac{2\cos 2x + 4}{\sin 2x}$ d) $y' = \frac{1}{x\cos 2x} - 4$

27. $y = 3^{\sin x}$ bo`lsa, u holda y' toping.

a) $y' = 3^{\sin x}\ln 3\cos x$ b) $y' = 3^{\sin x}\ln 3\,\text{tg}\, x$

c) $y' = 3^{\sin x}\cos x$ d) $y' = 3\cos x$

28. $y = 2\sin(2x+2) + \arcsin(2x+2)$ bo`lsa, u holda y' toping.

a) $y' = 4\cos(2x+2) - \frac{2}{\sqrt{1-(2x+2)^2}}$ b) $y' = 2\cos 2x + \frac{1}{\sqrt{1-4x^2}}$

c) $y' = 2\cos 2x + \frac{2}{\sqrt{1-4x^2}}$ d) $y' = \cos 2x - \frac{2}{\sqrt{1-4x^2}}$

29. $y = \text{tg}\, 3x + 5$ bo`lsa, u holda y' toping.

a) $y' = \frac{3}{\cos^2 x}$ b) $y' = -3\sin x$

c) $y' = -\frac{3}{\sin^2 x}$ d) $y' = 3\cos x$

30. $y = \text{ctg}\, 3x + 2$ bo`lsa, u holda y' toping.

a) $y' = \frac{3}{\cos^2 x}$ b) $y' = -3\sin x$

c) $y' = -\frac{3}{\sin^2 x}$ d) $y' = 3\cos x$

13-§. YUQORI TARTIBLI HOSILA VA DIFFERENSIALLAR. TEYLOR FORMULASI

13.1. Yuqori tartibli hosilalar (nazariy qism)

Differensiallanuvchi $y = f(x)$ funksiyani qaraymiz. Uning $f'(x)$ hosilasi ham x argumentnig funksiyasi bo`ladi. Agarda $\Delta x \to 0$ da $\frac{f'(x+\Delta x)-f'(x)}{\Delta x}$ nisbatning limiti mavjud bo`lsa, u holda $y = f(x)$ funksiya *ikkinchi (tartibli) hosilaga* ega deyiladi. Ikkinchi hosilalar y'', $f''(x)$, $\frac{d^2y}{dx^2}$ yoki $\frac{d^2f(x)}{dx^2}$ kabi belgilanadi.

Umuman olganda, ikkinchi hosiladan yana hosila olish mumkin. Bunda *uchinchi (tartibli) hosilaga* ega bo`lamiz: y''', $f'''(x)$, $\frac{d^3y}{dx^3}$, $\frac{d^3f(x)}{dx^3}$. Bu jarayonni davom ettirish mumkin.

$(n-1)$-tartibli hosiladan olingan hosila (mavjud bo`lgan taqdirda) *n-tartibli hosila* deyilladi. n-tartibli hosila $y^{(n)}$, $f^{(n)}(x)$, $\frac{d^ny}{dx^n}$ yoki $\frac{d^nf(x)}{dx^n}$ kabi belgilanadi.

$y = f(x)$ funksiyaning $f'(x)$ hosilasi *birinchi tartibli hosila*, $f''(x)$, $f'''(x)$, ..., $f^{(n)}(x)$, ..., hosilalari *yuqori tartibli hosilalar* deyiladi.

13.2. Yuqori tartibli differensiallar

Differensiallanuvchi $y = f(x)$ funksiyani qaraylik. Uning *birinchi tartibli differensiali* $dy = f'(x)dx$ bo`ladi.

Agar qaralayotgan funksiya ikki marta differensiallanuvchi bo`lsa, u holda
$d(dy) = d(f'(x)dx) = (f'(x)dx)'dx$ = ($(uv)' = u'v + uv'$ formulaga ko`ra) =
= $(f''(x)dx + f'(x)(dx)')dx$ = ($(dx)' = 0$ bo`lgani uchun) = $(f''(x)dx)dx$ =
= $f''(x)(dx)^2 = f''(x)dx^2$.

Eslatmalar. 1) Bu tengliklardagi $(dx)' = 0$ tenglik $dx = \Delta x$ orttirma x ga bog`liq emasligidan kelib chiqadi.

2) $(dx)^2 = dx^2$ belgilash kelishilgan, ya'ni dx^2 ifodadagi kvadrat x ga taalluqli bo`lmay, dx differensialning kvadratini anglatadi.

$d(dy)$ differensial *ikkinchi tartibli differensial* deb yuritiladi va d^2y kabi belgilanadi. Shunday qilib,

$$d^2y = f''(x)dx^2.$$

Agarda $y = x$ funksiya qaralsa, oxirgi tenglikdan quyidagi *muhim natija* kelib chiqadi: $d^2x = x''dx^2 = (x')'dx^2 = 1'dx^2 = 0$, ya'ni

$$d^2x = 0.$$

Agar qaralayotgan funksiya uch marta differensiallanuvchi bo`lsa, u holda *ikkinchi tartibli differensial*ni topish mumkin:

$$d^3y = f'''(x)dx^3.$$

Bu jarayonni davom ettirish mumkin.

$(n-1)$-tartibli differensialning differensiali *n-tartibli differensial* deyiladi va

$$d^ny = f^{(n)}(x)dx^n$$

kabi yoziladi. Aniqlanishiga ko`ra,

$$d^n y = d(d^{n-1}y).$$

$y = f(u)$ va $u = g(x)$ funksiyalar $y = y(x)$ murakkab funksiyani aniqlasin. U holda

$$d^2y = d(dy) = d(f'(u)du) = d\big(f'(u)\big)du + f'(u)d(du) =$$
$$= (f''(u)du)du + f'(u)d^2u = f''(u)du^2 + f'(u)d^2u,$$

ya'ni

$$d^2y = f''(u)du^2 + f'(u)d^2u.$$

Agarda $u = g(x)$ o`zgarmas yoki chiziqli bo`lmagan differensiallanuvchi funksiya bo`lsa, u holda $d^2u \neq 0$ bo`ladi.

Agar $y = f(u)$ funksiyada u argument erkli o`zgaruvchi bo`lsa, $d^2u = 0$ tenglikdan

$$d^2y = f''(u)du^2$$

ikkinchi tartibli differensialga ega bo`lamiz.

Shunday qilib, ikkinchi (va undan yuqori) tartibli differensiallarning shakli argumentga bog`liq ekan: agar $y = f(u)$ funksiyaning $u$ argumenti erkli o`zgaruvchi bo`lsa, ikkinchi tartibli differensial $d^2y = f''(u)du^2$ shaklda, agar $y = f(u)$ funksiyaning $u$ argumenti boshqa, aytaylik, $x$ argumentning funksiyasi bo`lsa, ikkinchi tartibli differensial $d^2y = f''(u)du^2 + f'(u)d^2u$ shaklda bo`ladi. Bu holat qisqacha **yuqori tartibli differensiallar invariantlik xossasiga ega emas**, deb bayon etiladi.

13.2.1.-misol. $y = 4x^5 - 7x^2 + 3$ funksiyaning dy, d^2y differensiallarini

$a)$ x – erkli o`zgaruvchi,

$b)$ x – boshqa erkli o`zgaruvchiga bog`liq funksiya

deb olib, toping.

Yechish. Birinchi tartibli dy differensial o`z shaklining invariantligi xossasidan ikkala $a)$ va $b)$ holda bir xil yoziladi:

$$dy = y'dx = (4x^5 - 7x^2 + 3)'dx = (20x^4 - 14x)dx.$$

Biroq, $a)$ holda dx deganda erkli x o`zgaruvchining $\Delta x$ orttirmasi tushuniladi ($dx = \Delta x$); $b)$ holda $dx$ deganda $x$ funksiyaning differensiali tushuniladi ($dx \neq \Delta x$ bo`lishi mumkin).

Yuqori tartibli differensiallar invariantlik xossasiga ega emasligidan d^2y differensialni izlashda ikkita holni alohida ko`rishga to`g`ri keladi.

$a)$ x – erkli o`zgaruvchi bo`lsin. U holda

$$d^2y = y''dx^2 = (y')'dx^2 = (20x^4 - 14x)'dx^2 = (80x^3 - 14)dx^2.$$

$b)$ x – boshqa erkli o`zgaruvchiga bog`liq funksiya bo`lsin. Bu holda $d^2y = f''(x)dx^2 + f'(x)d^2x$ formuladan ushbuga ega bo`lamiz

$$d^2y = (80x^3 - 14)dx^2 + (20x^4 - 14x)d^2x\ .$$

13.3. Teskari funksiyaning yuqori tartibli hosilalari

Agar $y = y(x)$ va $x = x(y)$ funksiyalar o`zaro teskari bo`lsa, y ni y ning "murakkab" funksiya deb qarash mumkin: $y = y\big(x(y)\big)$. Shuning uchun $v = v(u)$, $u = u(t)$ murakkab funksiyaning $v'_t = v'_u \cdot u'_t$ hosilasi formulasidan

$$y'_y = y'_x \cdot x'_y$$

tenglikka ega bo`lamiz. $y'_y = 1$ bo`lgani uchun teskari funksiyaning hosilasi

$$\boxed{x'_y = \frac{1}{y'_x}}$$

bo`ladi. $y'_x$ hosilani $y$ ning murakkab funksiyasi deb qarash mumkin: $y'_x = y'_x(x) = y'_x\big(x(y)\big)$. Murakkab funksiya hosilasi formulasini yana bir bor qo`llab

$$(y'_x)'_y = (y'_x)'_x \cdot x'_y = (y'_x)'_x \cdot \frac{1}{y'_x}$$

tenglikni hosil qilamiz. Bu tenglikni uchun ishlatib teskari funksiyaning ikkinchi tartibli hosilasi topiladi:

$$x''_{yy} = \left(x'_y\right)'_y = \left(\frac{1}{y'_x}\right)'_y = -\frac{(y'_x)'_y}{(y'_x)^2} = -\frac{y''_{xx}\cdot x'_y}{(y'_x)^2} = -\frac{y''_{xx}\cdot\frac{1}{y'_x}}{(y'_x)^2} = -\frac{y''_{xx}}{(y'_x)^3},$$

ya'ni

$$\boxed{x''_{yy} = -\frac{y''_{xx}}{(y'_x)^3}}$$

formulaga ega bo`lamiz.

Endi teskari funksiyaning uchinchi tartibli hosilasini topamiz. Bunda $y''_{xx} = y''_{xx}(x(y))$ deb olamiz va $(y''_{xx})'_y = (y''_{xx})'_x \cdot x'_y = (y''_{xx})'_x \cdot \frac{1}{y'_x}$ tenglikdan foydalanamiz:

$$x'''_{yyy} = \left(x''_{yy}\right)'_y = \left(-\frac{y''_{xx}}{(y'_x)^3}\right)'_y = -\frac{(y''_{xx})'_y\cdot(y'_x)^3 - y''_{xx}\cdot((y'_x)^3)'_y}{(y'_x)^6} =$$

$$= -\frac{y'''_{xxx}\cdot x'_y\cdot(y'_x)^3 - y''_{xx}\cdot 3(y'_x)^2\cdot y''_{xx}\cdot x'_y}{(y'_x)^6} = -\frac{y'''_{xxx}\cdot\frac{1}{y'_x}\cdot(y'_x)^3 - 3(y''_{xx})^2\cdot(y'_x)^2\cdot\frac{1}{y'_x}}{(y'_x)^6} =$$

$$= -\frac{y'''_{xxx}\cdot(y'_x)^2 - 3(y''_{xx})^2\cdot y'_x}{(y'_x)^6} = \frac{3(y''_{xx})^2 - y'''_{xxx}\cdot y'_x}{(y'_x)^5},$$

ya'ni

$$\boxed{x'''_{yyy} = \frac{3(y''_{xx})^2 - y'''_{xxx}\cdot y'_x}{(y'_x)^5}}.$$

Mazkur jarayonni davom ettirish mustaqil ishlashga havola etiladi.

13.3.1.-misol. Berilgan funksiyalarning ko`rsatilgan hosilalarini toping.

$$a)\ y = 3x - \frac{\cos x}{2};\ x''_{yy}; \qquad b)\ y = x + \mathrm{e}^x;\quad x'''_{yyy}.$$

Yechish. $a)$ Avval $y = 3x - \frac{\cos x}{2}$ funksiyaning x bo`yicha birinchi va ikkinchi tartibli hosilalarini topamiz:

$$y'_x = \left(3x - \frac{\cos x}{2}\right)'_x = 3 + \frac{\sin x}{2}, \quad y''_{xx} = (y'_x)'_x = \left(3 + \frac{\sin x}{2}\right)'_x = \frac{\cos x}{2}.$$

Keyin hosil bo`lgan bu ifodalarni $x''_{yy} = -\frac{y''_{xx}}{(y'_x)^3}$ formulaga qo`yamiz:

$$x''_{yy} = -\frac{\frac{\cos x}{2}}{\left(3+\frac{\sin x}{2}\right)^3} = -\frac{4\cos x}{(\sin x+6)^3}.$$

$b)$ $y = x + \mathrm{e}^x$ funksiyaning x bo`yicha birinchi, ikkinchi va uchinchi tartibli hosilalarini topamiz:

$$y'_x = (x + \mathrm{e}^x)'_x = 1 + \mathrm{e}^x,$$
$$y''_{xx} = (1 + \mathrm{e}^x)'_x = \mathrm{e}^x,$$
$$y'''_{xxx} = (\mathrm{e}^x)'_x = \mathrm{e}^x.$$

Hosil bo`lgan bu ifodalarni $x'''_{yyy} = \frac{3(y''_{xx})^2 - y'''_{xxx}\cdot y'_x}{(y'_x)^5}$ formulaga qo`yamiz:

$$x'''_{yyy} = \frac{3(\mathrm{e}^x)^2 - \mathrm{e}^x\cdot(1+\mathrm{e}^x)}{(1+\mathrm{e}^x)^5} = \frac{2\mathrm{e}^{2x} - \mathrm{e}^x}{(\mathrm{e}^x+1)^5}.$$

13.4. Oshkormas berilgan funksiyaning yuqori tartibli hosilalari

Agar $y = y(x)$ funksiya $F(x,\ y) = 0$ tenglama bilan oshkormas berilgan bo`lsa, birinchi tartibli $y'_x$ hosilani quyidagicha topish mumkin: dastlab $y$ ni $x$ ning funksiyasi deb $F(x,\ y) = 0$ tenglama $x$ bo`yicha differensiallanadi, keyin hosil bo`lgan $F'_x(x,\ y) = 0$ tenglama y'_x hosilaga nisbatan yechiladi.

y''_{xx} hosilani topish uchun $F(x,\ y) = 0$ tenglama x bo`yicha ikki marta differensiallanadi va hosil bo`lgan $F''_{xx}(x,\ y) = 0$ tenglama y''_{xx} hosilaga nisbatan yechiladi. Shunday usulda y'''_{xxx}, y^{IV}_{xxxx} va hakozo hosilalar topiladi.

13.4.1.-misol. y''_{xx} hosilani toping:

$a)$ $\operatorname{arctg} y - y + x = 0$; $\quad b)$ $\mathrm{e}^x - \mathrm{e}^y = y - x$; $\quad c)$ $x + y = \mathrm{e}^{x-y}$.

Yechish. $a)$ y ni x ning funksiyasi deb, $\operatorname{arctg} y - y + x = 0$ tenglamadan x bo`yicha hosila olinadi:

$$\frac{y'_x}{1+y^2} - y'_x + 1 = 0.$$

Hosil bo`lgan bu tenglamani y'_x ga nisbatan yechiladi:

$$y'_x\left(\frac{1}{1+y^2} - 1\right) + 1 = 0, \quad y'_x \cdot \frac{-y^2}{1+y^2} = -1, \quad y'_x = \frac{1+y^2}{y^2} = y^{-2} + 1.$$

Olingan $y'_x = y^{-2} + 1$ funksiya x bo`yicha differensiallaniladi:

$$y''_{xx} = -2y^{-3}y'_x = -\frac{2}{y^3}\cdot y'_x.$$

Bu yerga y'_x ning topilgan qiymati qo`yilib, so`raglan ushbu hosilaga erishiladi

$$y''_{xx} = -\frac{2}{y^3}\cdot\frac{1+y^2}{y^2} = -\frac{2(y^2+1)}{y^5}.$$

$b)$ Yuqoridagilarga o`xshash amallarni bajarib, quyidagilar hosil qilinadi:

$$(\mathrm{e}^x - \mathrm{e}^y)'_x = (y - x)'_x, \qquad \mathrm{e}^x - \mathrm{e}^y \cdot y'_x = y'_x - 1,$$
$$(\mathrm{e}^y + 1)\cdot y'_x = \mathrm{e}^x + 1, \qquad y'_x = \frac{\mathrm{e}^x+1}{\mathrm{e}^y+1}.$$

$$y''_{xx} = (y'_x)'_x = \left(\frac{\mathrm{e}^x+1}{\mathrm{e}^y+1}\right)'_x = \frac{\mathrm{e}^x(\mathrm{e}^y+1) - \mathrm{e}^y\cdot y'_x(\mathrm{e}^x+1)}{(\mathrm{e}^y+1)^2} = \frac{\mathrm{e}^x(\mathrm{e}^y+1) - \mathrm{e}^y(\mathrm{e}^x+1)\cdot\frac{\mathrm{e}^x+1}{\mathrm{e}^y+1}}{(\mathrm{e}^y+1)^2} =$$

$$= \frac{e^x(e^y+1)^2 - e^y(e^x+1)^2}{(e^y+1)^3},$$

ya'ni $y''_{xx} = \frac{e^x(e^y+1)^2 - e^y(e^x+1)^2}{(e^y+1)^3}$.

$c)$ Analogik amallar bajariladi:

$$(x+y)'_x = (e^{x-y})'_x, \qquad 1 + y'_x = e^{x-y}\cdot(1-y'_x),$$

$$y'_x(e^{x-y}+1) = e^{x-y} - 1, \qquad y'_x = \frac{e^{x-y}-1}{e^{x-y}+1}.$$

$$y''_{xx} = (y'_x)'_x = \left(\frac{e^{x-y}-1}{e^{x-y}+1}\right)'_x = \frac{e^{x-y}(1-y'_x)(e^{x-y}+1) - e^{x-y}(1-y'_x)(e^{x-y}-1)}{(e^{x-y}+1)^2} = \frac{2e^{x-y}(1-y'_x)}{(e^{x-y}+1)^2} =$$

$$= \frac{2e^{x-y}\left(1-\frac{e^{x-y}-1}{e^{x-y}+1}\right)}{(e^{x-y}+1)^2} = \frac{4e^{x-y}}{(e^{x-y}+1)^3},$$

ya'ni $y''_{xx} = \frac{4e^{x-y}}{(e^{x-y}+1)^3}$.

13.5. Parametrik berilgan funksiyaning yuqori tartibli hosilalari

Agar $y = y(x)$ funksiya parametrik berilgan bo`lsa: $x = \varphi(t)$, $y = \psi(t)$, u holda birinchi tartibli y'_x hosila

$$y'_x = \frac{dy}{dx} = \frac{y'_t}{x'_t}$$

formula bilan topiladi. Parametrik usulda berilgan funksiyaning yuqori tartibli hosilalari ketma-ket quyidagicha topiladi:

$$y''_{xx} = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d(y'_x)}{dx} = \frac{\frac{d(y'_x)}{dt}}{\frac{dx}{dt}} = \frac{(y'_x)'_t}{x'_t},$$

ya'ni

$$\boxed{y''_{xx} = \frac{(y'_x)'_t}{x'_t}};$$

$$y'''_{xxx} = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d\left(\frac{d^2y}{dx^2}\right)}{dx} = \frac{d(y''_{xx})}{dx} = \frac{\frac{d(y''_{xx})}{dt}}{\frac{dx}{dt}} = \frac{(y''_{xx})'_t}{x'_t},$$

ya'ni

$$\boxed{y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t}};$$

va hakozo

$$\boxed{y^{(n)}_{x\ldots x} = \frac{\left(y^{(n-1)}_{x\ldots x}\right)'_t}{x'_t}}.$$

Amaliy masalalarni yechishda quyidagi formulalar qo`llaniladi:

Ikkinchi tartibli hosila uchun $y''_{xx} = \frac{(y'_x)'_t}{x'_t} = \frac{\left(\frac{y'_t}{x'_t}\right)'_t}{x'_t} = \frac{\frac{y''_{tt}x'_t - y'_t x''_{tt}}{(x'_t)^2}}{x'_t} = \frac{y''_{tt}x'_t - y'_t x''_{tt}}{(x'_t)^3}$,

ya'ni

$$\boxed{y''_{xx} = \frac{y''_{tt}x'_t - y'_t x''_{tt}}{(x'_t)^3}};$$

Uchinchi tartibli hosila uchun

$$y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t} = \frac{\left(\frac{y''_{tt}x'_t - y'_t x''_{tt}}{(x'_t)^3}\right)'_t}{x'_t} =$$

$$= \frac{\frac{(y'''_{ttt}x'_t + y''_{tt}x''_{tt} - y''_{tt}x''_{tt} - y'_t x'''_{ttt})(x'_t)^3 - 3(x'_t)^2 x''_{tt}(y''_{tt}x'_t - y'_t x''_{tt})}{(x'_t)^6}}{x'_t} =$$

$$= \frac{(y'''_{ttt}x'_t - y'_t x'''_{ttt})x'_t - 3x''_{tt}(y''_{tt}x'_t - y'_t x''_{tt})}{(x'_t)^5},$$

ya'ni

$$\boxed{y'''_{xxx} = \frac{(y'''_{ttt}x'_t - y'_t x'''_{ttt})x'_t - 3x''_{tt}(y''_{tt}x'_t - y'_t x''_{tt})}{(x'_t)^5}}.$$

13.5.1.-misol. Parametrik usulda berilgan y funksiyaning x bo`yicha ikkinchi tartibli hosilasini toping.

$a)$ $x = a\cos^3 t$, $y = b\sin^3 t$; $b)$ $x = t^3 + 3t + 1$, $y = t^3 - 3t + 1$;
$c)$ $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$;
$d)$ $x = \mathrm{e}^t \cos t$, $y = \mathrm{e}^t \sin t$.

Yechish. $a)$ Dastlab x'_t va y'_t hosilalar topiladi:

$$x'_t = (a\cos^3 t)'_t = -3a\cos^2 t\sin t, \quad y'_t = (b\sin^3 t)'_t = 3b\sin^2 t\cos t.$$

Keyin $y'_x = \frac{y'_t}{x'_t}$ formula ishlatiladi va y funksiyaning x bo`yicha birinchi tartibli hosilasi topiladi:

$$y'_x = \frac{3b\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{b}{a}\operatorname{tg} t, \qquad \left(t \neq \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}\right).$$

Endi $y''_{xx} = \frac{(y'_x)'_t}{x'_t}$ yordamida y''_{xx} hosila izlanadi:

$$(y'_x)'_t = \left(-\frac{b}{a}tg\ t\right)'_t = -\frac{b}{a\cos^2 t}, \qquad y''_{xx} = \frac{-\frac{b}{a\cos^2 t}}{-3a\cos^2 t\sin t} = \frac{b}{3a^2\cos^4 t\sin t}.$$

$b)$ x'_t, x''_{tt}, y'_t va y''_{tt} hosilalar topiladi:

$$x'_t = (t^3 + 3t + 1)'_t = 3t^2 + 3, \qquad x''_{tt} = (x'_t)'_t = (3t^2 + 3)'_t = 6t;$$
$$y'_t = (t^3 - 3t + 1)'_t = 3t^2 - 3, \qquad y''_{tt} = (y'_t)'_t = (3t^2 - 3)'_t = 6t;$$

$y''_{xx} = \frac{y''_{tt}x'_t - y'_t x''_{tt}}{(x'_t)^3}$ formulani qo`llab, ushbu topiladi

$$y''_{xx} = \frac{6t(3t^2+3) - 6t(3t^2-3)}{(3t^2+3)^3} = \frac{36t}{(3t^2+3)^3}.$$

13.5.2.-misol. Ushbu misollarda y'''_{xxx} ni toping:

$a)$ $x = \mathrm{e}^{-t}$, $y = t^3$; $b)$ $x = \sec t$, $y = \operatorname{tg} t$.

Yechish. $a)$ x'_t va y'_t hosilalar topiladi:

$$x'_t = -\mathrm{e}^{-t}, \qquad y'_t = 3t^2.$$

Bularni $y'_x = \frac{y'_t}{x'_t}$ formulaga qo`yib, ushbu olinadi:

$$y'_x = -\frac{3t^2}{e^{-t}}.$$

$y''_{xx} = \frac{(y'_x)'_t}{x'_t}$ formulani tadbiq etib, ikkinchi tartibli hosila topiladi:

$$y''_{xx} = \frac{\left(-\frac{3t^2}{e^{-t}}\right)'_t}{-e^{-t}} = \frac{\frac{6te^{-t}+3t^2e^{-t}}{e^{-2t}}}{e^{-t}} = \frac{3t(t+2)}{e^{-2t}} = 3t(t+2)e^{2t}.$$

Nihoyat $y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t}$ formulani ishlatib, talab etilgan hosilaga erishiladi:

$$y'''_{xxx} = \frac{(3t(t+2)e^{2t})'_t}{-e^{-t}} = \frac{3(t+2)e^{2t} + 3te^{2t} + 6t(t+2)e^{2t}}{-e^{-t}} =$$

$$= \frac{e^{2t}(6t^2+18t+6)}{-e^{-t}} = -6e^{3t}(t^2+3t+1),$$

ya'ni $y'''_{xxx} = -6e^{3t}(t^2+3t+1)$.

b) Yuqoridagilarga o`xshash amallarni bajarib, quyidagilar hosil qilinadi:

$$x'_t = (\sec t)'_t = \left(\frac{1}{\cos t}\right)'_t = \frac{\sin t}{\cos^2 t},$$

$$y'_t = (\operatorname{tg} t)'_t = \frac{1}{\cos^2 t}. \qquad y'_x = \frac{y'_t}{x'_t} = \frac{\frac{1}{\cos^2 t}}{\frac{\sin t}{\cos^2 t}} = \frac{1}{\sin t}.$$

$$y''_{xx} = \frac{(y'_x)'_t}{x'_t} = \frac{\left(\frac{1}{\sin t}\right)'_t}{x'_t} = \frac{-\frac{\cos t}{\sin^2 t}}{\frac{\sin t}{\cos^2 t}} = -\frac{\cos^3 t}{\sin^3 t} = -\operatorname{ctg}^3 t.$$

$$y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t} = \frac{(-\operatorname{ctg}^3 t)'_t}{\frac{\sin t}{\cos^2 t}} = \frac{3\operatorname{ctg}^2 t \cdot \frac{1}{\sin^2 t}}{\frac{\sin t}{\cos^2 t}} = 3\frac{\cos^4 t}{\sin^5 t},$$

ya'ni $y'''_{xxx} = 3\frac{\cos^4 t}{\sin^5 t}$.

13.6. Teylor formulasi

Ma'lumki, qiymatlarini hisoblash ma'nosida soddaroq funksiyalar bu – ko`phadlardir. $f(x)$ funksiyani $x_0$ nuqta atrofida qandaydir darajali ko`phad bilan almashtirish mumkinligi to`g`risida savol paydo bo`ladi. Bu savolga matematik analizda muhim formulalardan biri bo`lgan Teylor formulasi javob beradi. Bu formula ko`plab nazariy va amaliy tadbiqlarga ega va taqribiy hisoblashlarning asosi hisoblanadi. Mazkur formulani keltiramiz.

Aytaylik, $f(x)$ funksiya x_0 nuqta biror atrofida 1-, 2-, ..., $(n+1)$-tartibli hosilalarga ega bo`lsin. U holda bu atrofning har bir $x$ nuqtasi uchun *Teylor formulasi* deb ataluvchi ushbu tenglik o`rinli bo`ladi

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots +$$

$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_{n+1}(x)\ .$$

Bu yerda $R_{n+1}(x)$ – *qoldiq had*, chunki bunday ko`phad funksiya bilan faqatgina taqribiy almashtirilishi mumkin. Qoldiq had turlicha yozilishi mumkin. Qoldiq hadning *Lagranj shaklidagi qoldiq had* deb ataluvchi ushbu ko`rinishini qaraylik

$$R_{n+1}(x) = \frac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1}\ , \quad \theta \in (0,\ 1).$$

Agar bu formulada $x_0 = 0$ deb olinsa, u holda ushbu *Makloren formulasi* hosil bo`ladi:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}.$$

Bunda $\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$ – Lagranj shaklidagi qoldiq had. Shunday qilib, Makloren formulasi Teylor formulasi ning xususiy holi ekan.

Teylor formulasining o`ng tomonidagi yig`indi funksiyaning Teylor formulasi bo`ycha *yoyilmasi* deyiladi. Funksiyadan uning yoyilmasiga o`tish jarayoni *funksiyani (Teylor formulasi bo`yicha) yoyish* deb yuritiladi.

13.6.1. e^x, $\sin x$, $\cos x$, $(1+x)^\alpha$, $\ln(1+x)$ funksiyalarni Teylor formulasi bo`ycha yoyish

1. $f(x) = \mathrm{e}^x$ funksiya barcha $x \in (-\infty, +\infty)$ nuqtalarda ixtiyoriy tartibli hosilaga ega bo`lgani uchun bu nuqtalarda mazkur funksiya uchun Teylor formulasini qo`llash mumkin.

$f(x) = \mathrm{e}^x$ funksiyani $x_0 = 0$ nuqta atrofida yoyamiz. Buning uchun uning $(n+1)$-tartibligacha hosilalarni olamiz:

$$f'(x) = \mathrm{e}^x,$$
$$f''(x) = \mathrm{e}^x,$$
$$\dots$$
$$f^{(n)}(x) = \mathrm{e}^x,$$
$$f^{(n+1)}(x) = \mathrm{e}^x.$$

Bu hosilalarning $x_0 = 0$ nuqtadagi qiymatlarini hisoblaymiz:

$$f(0) = \mathrm{e}^0 = 1,$$
$$f'(0) = \mathrm{e}^0 = 1,$$
$$f''(0) = \mathrm{e}^0 = 1,$$
$$\dots$$
$$f^{(n)}(0) = \mathrm{e}^0 = 1,$$
$$f^{(n+1)}(\theta x) = \mathrm{e}^{\theta x}, \quad \theta \in (0,\ 1).$$

Hosil bo`lgan bu qiymatlarni Makloren formulasiga qo`yib, $f(x) = \mathrm{e}^x$ funksiyaning nolning atrofidagi ushbu yoyilmasiga ega bo`lamiz:

$$\boxed{\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\theta x}}, \quad \theta \in (0,\ 1).$$

2. $f(x) = \sin x$ funksiya barcha $x \in (-\infty, +\infty)$ nuqtalarda ixtiyoriy tartibli hosilaga ega bo`lgani uchun bu nuqtalarda mazkur funksiya uchun Teylor formulasini qo`llash mumkin.

$f(x) = \sin x$ funksiyani $x_0 = 0$ nuqta atrofida yoyamiz. Buning uchun uning $(n+1)$-tartibligacha hosilalarni olamiz:

$$f'(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$
$$f''(x) = -\sin x = \sin(x + \pi) = \sin\left(x + \frac{2\pi}{2}\right),$$

$$f'''(x) = -\cos x = \sin\left(x + \frac{3\pi}{2}\right),$$
$$f^{\text{iv}}(x) = \sin x = \sin(x + 2\pi) = \sin\left(x + \frac{4\pi}{2}\right),$$
$$\dots$$
$$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right).$$

Bu hosilalarning $x_0 = 0$ nuqtadagi qiymatlarini hisoblaymiz:

$$f(0) = \sin 0 = 0,$$
$$f'(0) = \cos 0 = 1,$$
$$f''(0) = -\sin 0 = 0,$$
$$\dots$$
$$f^{(n)}(0) = \sin\frac{n\pi}{2},$$
$$f^{(n+1)}(\theta x) = \sin\left(\theta x + \frac{(n+1)\pi}{2}\right), \quad \theta \in (0,\ 1).$$

Hosil bo`lgan bu qiymatlarni Makloren formulasiga qo`yib, $f(x) = \sin x$ funksiyaning nolning atrofidagi ushbu yoyilmasiga ega bo`lamiz:

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{x^n}{n!}\sin\frac{n\pi}{2} + \frac{x^{n+1}}{(n+1)!}\sin\left(\theta x + \frac{(n+1)\pi}{2}\right)},$$
$$\theta \in (0,\ 1).$$

3. Xuddi shunday usul bilan $\cos x$ funksiya nolning atrofida Teylor formulasi bo`yicha quyidagicha yoyiladi:

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{x^n}{n!}\cos\frac{n\pi}{2} + \frac{x^{n+1}}{(n+1)!}\cos\left(\theta x + \frac{(n+1)\pi}{2}\right)},$$
$$\theta \in (0,\ 1).$$

4. $f(x) = (1 + x)^\alpha$ funksiya barcha $x \in (-\infty, +\infty)$ nuqtalarda ixtiyoriy tartibli hosilaga ega bo`lgani uchun bu nuqtalarda mazkur funksiya uchun Teylor formulasini qo`llash mumkin.

$f(x) = (1 + x)^\alpha$ funksiyani $x_0 = 0$ nuqta atrofida yoyamiz. Buning uchun uning $(n + 1)$-tartibligacha hosilalarni olamiz:

$$f'(x) = \alpha(1 + x)^{\alpha-1},$$
$$f''(x) = \alpha(\alpha - 1)(1 + x)^{\alpha-2},$$
$$\dots$$
$$f^{(n)}(x) = \alpha(\alpha - 1)\dots(\alpha - n + 1)(1 + x)^{\alpha-n},$$
$$f^{(n+1)}(x) = \alpha(\alpha - 1)\dots(\alpha - n + 1)(\alpha - n)(1 + x)^{\alpha-n-1}.$$

Bu hosilalarning $x_0 = 0$ nuqtadagi qiymatlarini hisoblaymiz:

$$f'(0) = \alpha,$$
$$f''(0) = \alpha(\alpha - 1),$$
$$\dots$$
$$f^{(n)}(0) = \alpha(\alpha - 1)\dots(\alpha - n + 1),$$
$$f^{(n+1)}(\theta x) = \alpha(\alpha - 1)\dots(\alpha - n + 1)(\alpha - n)(1 + \theta x)^{\alpha-n-1},$$
$$\theta \in (0,\ 1).$$

Hosil bo`lgan bu qiymatlarni Makloren formulasiga qo`yib, $f(x) = (1 + x)^\alpha$ funksiyaning nolning atrofidagi ushbu yoyilmasiga ega bo`lamiz:

$$\boxed{(1+x)^{\alpha} = 1 + \frac{x}{1!}\alpha + \frac{x^2}{2!}\alpha(\alpha-1) + \cdots + \frac{x^n}{n!}\alpha(\alpha-1)\ldots(\alpha-n+1) +}$$
$$\boxed{+\frac{x^{n+1}}{(n+1)!}\alpha(\alpha-1)\ldots(\alpha-n+1)(\alpha-n)(1+\theta x)^{\alpha-n-1}},$$
$$\theta \in (0,\ 1).$$

5. $f(x) = \ln(1+x)$ funksiyani nolning atrofida Teylor formulasi bo`yicha yoyamiz:

$$f(x) = \ln(1+x), \qquad f(0) = 0,$$
$$f'(x) = \frac{1}{1+x}, \qquad f'(0) = 1,$$
$$f''(x) = -\frac{1}{(1+x)^2}, \qquad f''(0) = -1,$$
$$f'''(x) = \frac{1\cdot 2}{(1+x)^3}, \qquad f'''(0) = 1\cdot 2 = 2!,$$
$$f^{\text{iv}}(x) = -\frac{1\cdot 2\cdot 3}{(1+x)^4}, \qquad f^{\text{iv}}(0) = -1\cdot 2\cdot 3 = -3!,$$
$$\ldots, \qquad \ldots$$
$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}, \qquad f^{(n)}(0) = (-1)^{n-1}(n-1)!,$$
$$f^{(n+1)}(x) = (-1)^n\frac{n!}{(1+x)^{n+1}}, \qquad f^{(n+1)}(\theta x) = (-1)^n\frac{n!}{(1+\theta x)^{n+1}},$$
$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n\frac{x^{n+1}}{n(1+\theta x)^{n+1}}},$$
$$\theta \in (0,\ 1).$$

13.6.2. Teylor formulasining taqribiy hisoblashlarga tadbiqi

13.6.2.1.-misol. Teylor formulasidan foydalanib, $P(x) = x^5 - 2x^4 + x^3 - x^2 + 2x - 1$ ko`phadni $x - 1$ ikkihad darajalari bo`yicha yoying.

Yechish. Masalani yechish uchun avvalo ushbu hosilalarni topamiz:

$$P'(x) = 5x^4 - 8x^3 + 3x^2 - 2x + 2,$$
$$P''(x) = 20x^3 - 24x^2 + 6x - 2,$$
$$P'''(x) = 60x^2 - 48x + 6,$$
$$P^{\text{iv}}(x) = 120x - 48,$$
$$P^{\text{v}}(x) = 120,$$
$$P^{\text{vi}}(x) = 0,$$
$$\ldots$$
$$P^{(n)}(x) = 0, n \geq 6.$$

Endi ko`phad va uning hosilalarining $x = 1$ nuqtadagi qiymatlarini hisoblaymiz:

$$P(1) = 1^5 - 2\cdot 1^4 + 1^3 - 1^2 + 2\cdot 1 - 1 = 0,$$
$$P'(1) = 5\cdot 1^4 - 8\cdot 1^3 + 3\cdot 1^2 - 2\cdot 1 + 2 = 0,$$
$$P''(1) = 20\cdot 1^3 - 24\cdot 1^2 + 6\cdot 1 - 2 = 0,$$
$$P'''(1) = 60\cdot 1^2 - 48\cdot 1 + 6 = 18,$$
$$P^{\text{iv}}(1) = 120\cdot 1 - 48 = 72,$$

$P^{\text{v}}(1) = 120,$
$P^{\text{vi}}(1) = 0,$
$\dots$
$P^{(n)}(1) = 0,\ n \geq 6.$

Topilgan qiymatlarni Teylor formulasiga qo`yib, ushbuga ega bo`lamiz:

$$P(x) = \frac{18}{3!}(x-1)^3 + \frac{72}{4!}(x-1)^4 + \frac{120}{5!}(x-1)^5.$$

$3! = 1 \cdot 2 \cdot 3 = 6,\ 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ va $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$ ekanligidan, oxirgi ko`rinish

$$P(x) = 3(x-1)^3 + 3(x-1)^4 + (x-1)^5$$

kabi bo`ladi.

13.6.2.2.-misol. $[0,\ 1]$ kesmada berilgan $f(x) = \ln(1+x)$ funksiya Makloren formulasidan foydalanib, x ning darajalari bo`yicha yoyilsin. Dastlabki o`nta had qoldirilgandagi xatolik topilsin.

Yechish. $f(x) = \ln(1+x)$ funksiyani nolning atrofida yoyamiz:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n \frac{x^{n+1}}{(n+1)(1+\theta x)^{n+1}},$$
$$\theta \in (0,\ 1).$$

$n + 1 = 10$ bo`lgani uchun

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{x^9}{9} - \frac{x^{10}}{10(1+\theta x)^{10}}.$$

Qoldiq had $R_{10}(x) = -\frac{x^{10}}{10(1+\theta x)^{10}}$ ning absolyut qiymatini baholaymiz. Bunda $0 \leq x \leq 1$ va $0 < \theta < 1$, natijada, $0 \leq \theta x < 1$ ekanligini e'tiborga olib, ushbuga ega bo`lamiz

$$|R_{10}(x)| = \left|-\frac{x^{10}}{10(1+\theta x)^{10}}\right| = \frac{x^{10}}{10(1+\theta x)^{10}} < \frac{1}{10}.$$

13.6.2.3.-misol. $f(x) = \mathrm{e}^x$ funksiyani $[-1,\ 1]$ kesmada 0,001 aniqlikda ifodalaydigan ko`phad hosil qilish uchun Makloren formulasida nechta had olish kerak?

Yechish. $f(x) = \mathrm{e}^x$ funksiyaning Makloren formulasi

$$\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\theta x}, \qquad \theta \in (0,\ 1)$$

ko`rinishda bo`ladi. Qoldiq had $R_{n+1}(x) = \frac{x^{n+1}}{(n+1)!}\mathrm{e}^{\theta x}$ ning absolyut qiymatini baholaymiz. Shartga ko`ra $|x| \leq 1$ va $0 < \theta < 1$ bo`lgani uchun

$$|R_{n+1}(x)| = \left|\frac{x^{n+1}}{(n+1)!}\mathrm{e}^{\theta x}\right| < \frac{|x|^{n+1}}{(n+1)!}\mathrm{e}^{\theta x} < \frac{1}{(n+1)!}\mathrm{e} < \frac{3}{(n+1)!}$$

bo`ladi. Natijada $\frac{3}{(n+1)!} \leq 0{,}001$ tengsizlik bajarilishidan $|R_{n+1}(x)| \leq 0{,}001$ tengsizlik kelib chiqadi. $\frac{3}{(n+1)!} \leq 0{,}001$ tengsizlikdan $(n+1)! \geq \frac{3}{0{,}001} = 3000$ kelib chiqadi. $7! = 5040$ bo`lganligidan $n + 1 \geq 7$ deb olish kifoya.

Javob: Shunday qilib, Makloren formulasida 7 had olish yetarli ekan.

13.6.2.4.-misol. x ning qanday qiymatlarida

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

taqribiy tenglikning xatoligi 0,00005 dan kichik bo`ladi?

Yechish. Berilgan taqribiy tenglikning o`ng tomoni

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{x^n}{n!}\cos\frac{n\pi}{2} + \frac{x^{n+1}}{(n+1)!}\cos\left(\theta x + \frac{(n+1)\pi}{2}\right),\ \ \theta \in (0,\ 1)$$

Makloren formulasining dastlabki olti hadidan iborat (ikkinchi, to`rtinchi va oltinchi hadlar nolga teng). Shuning uchun $R_6(x) = \frac{x^6}{6!}\cos\left(\theta x + \frac{6\pi}{2}\right) = -\frac{x^6}{6!}\cos\theta x$ qoldiq hadni baholaymiz:

$$|R_6(x)| = \left|-\frac{x^6}{6!}\cos\theta x\right| \le \frac{|x|^6}{6!}.$$

Xatolik 0,00005 dan kichik bo`lishi uchun x ning qiymatlarini $\frac{|x|^6}{6!} < 0{,}00005$ tengsizlikni qanoatlantiradigan qilib olamiz. Bu tengsizlikni yechib, $|x| < 0{,}575$, yoki $x \in (-0{,}575;\ \ 0{,}575)$ ni hosil qilamiz.

Javob: x ning $(-0{,}575;\ \ 0{,}575)$ intervaldagi qiymatlarida berilgan taqribiy tenglikning xatoligi 0,00005 dan kichik bo`ladi.

13.6.2.5.-Misol. Quyidagi ifodalarning taqribiy qiymatlarni 10^{-5} aniqlikda hisoblang:

$$a)\ \cos 5^\circ; \qquad b)\ \sin 20^\circ.$$

Yechish. $a)$ Ushbu Makloren formulasiga

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{x^n}{n!}\cos\frac{n\pi}{2} + \frac{x^{n+1}}{(n+1)!}\cos\left(\theta x + \frac{(n+1)\pi}{2}\right),\ \ \theta \in (0,\ 1)$$

$x = 5^\circ = \frac{\pi}{36}$ qiymatni qo`yamiz.

$$\frac{x^2}{2!} = \frac{\pi^2}{2\cdot 36^2} \approx 0{,}0038, \qquad \frac{x^4}{4!} = \frac{\pi^4}{24\cdot 36^4} \approx 2{,}4\cdot 10^{-6},$$

bo`lgani uchun ushbu hadlar bilan kifoyalanamiz: $\cos x \approx 1 - \frac{x^2}{2!}$.

Bunda xatolik quyidagicha baholanadi

$$\left|R_4\left(\frac{\pi}{36}\right)\right| = \left|\frac{\left(\frac{\pi}{36}\right)^4}{4!}\cos\left(\theta\cdot\frac{\pi}{36} + \frac{4\pi}{2}\right)\right| \le \frac{\left|\frac{\pi}{36}\right|^4}{4!} \approx 2{,}4\cdot 10^{-6} < 2{,}5\cdot 10^{-6}.$$

Shunday qilib, talab etilgan aniqlik bilan ushbuga ega bo`lamiz

$$\cos 5^\circ = \cos\frac{\pi}{36} \approx 1 - \frac{\left(\frac{\pi}{36}\right)^2}{2!} \approx 1 - 0{,}0038 = 0{,}9962\ .$$

13.6.2.6.-Misol. $x > 0$ bo`lganda $x - \frac{x^2}{2} < \ln(1-x) < x$ tengsizlikni isbotlang.

Yechish. $R_2(x)$ qoldiq hadli Makloren formulasi ushbu ko`rinishda bo`ladi:

$$\ln(1+x) = x - \frac{x^2}{2!(1+\theta x)^2}, \qquad \theta \in (0,\ 1).$$

$R_3(x)$ qoldiq hadli Makloren formulasi esa ushbu ko`rinishda bo`ladi:

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!(1+\theta_1 x)^3}, \qquad \theta_1 \in (0,\ 1).$$

$\frac{x^2}{2!(1+\theta x)^2} > 0$ bo`lgani uchun $\ln(1+x) = x - \frac{x^2}{2!(1+\theta x)^2} < x$, ya'ni

$$\ln(1+x) < x$$

bo`ladi. $x > 0$ da $\frac{x^3}{3!(1+\theta_1 x)^3} > 0$ ekanligidan $\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!(1+\theta_1 x)^3} > x - \frac{x^2}{2!}$, ya'ni

$$\ln(1+x) > x - \frac{x^2}{2}$$

kelib chiqadi. Bu ikkala tengsizlikdan

$$x - \frac{x^2}{2} < \ln(1+x) < x$$

hosil bo`ladi.

Mustahkamlash uchun mashqlar

Yuqori tartibi hosilalarga oid masalalar

1. Berilgan funksiyaning y' va y'' hosilasini toping.

1. $y = x + arctg 2x$
2. $y = 25x - \cos^2 x$
3. $y^2 = 8x$
4. $\frac{x^2}{5} + \frac{y^2}{7} = 1$
5. $y = x + arctg\, y$
6. $\frac{x^2}{5} + \frac{y^2}{3} = 1$
7. $y^2 = 25x - 4$
8. $arctg\, y = 3x + 5y$
9. $tgy + 5x - 4y = 0$
10. $e^y = 2y + 2x$

2. Berilgan funksiyaning y' va y'' hosilasini toping.

1. $\begin{cases} y = t^3 + t^2 - 1 \\ y = t^2 + t + 1 \end{cases}$
2. $\begin{cases} y = 2sin^3 t \\ x = 2cos^3 t \end{cases}$
3. $\begin{cases} y = (2t+3)sin^2 t \\ x = 3t^2 \end{cases}$
4. $\begin{cases} y = 6sin^3 t \\ x = 2cos^3 t + 3 \end{cases}$
5. $\begin{cases} y = e^{4t} \\ x = e^{-2t} \end{cases}$
6. $\begin{cases} y = \sqrt{t} \\ x = \sqrt[5]{t} \end{cases}$

3. Berilgan funksiyaning $y^{(n)}$ hosilasini toping.

1. $y = \ln x$
2. $y = \frac{1}{x}$
3. $y = \cos x$
4. $y = \sin x$
5. $y = \frac{1}{x+5}$
6. $y = e^{-2x}$
7. $y = \ln(3+x)$
8. $y = 2^x$

O`z-o`zini tekshirish uchun savollar

1. Yuqori tartibli hosilani tushuntiring.
2. Yuqori tartibli differensialni tushuntiring.
3. Teskari funksiyaning yuqori tartibli hosilalarini topish formulalarini yozing.
4. Oshkormas ko`rinishida berilgan funksiyaning yuqori tartibli hosilalarini topish formulasini yozing.
5. Parametrik ko`rinishida berilgan funksiyaning yuqori tartibli hosilalarini topish formulasini yozing.
6. Teylor formulasi yozing.
7. Teylor formulasining taqribiy hisoblashlarga tadbiqini tushuntiring.

O`z-o`zini tekshirish uchun testlar

Yuqori tartibi hosilalarga oid testlar

1. Hosila bu …
a) funksiya b) urinma c) grafik d) differentsial
2. $y = f(x)$ funksiyaning $f'(x)$ hosilasidan olingan hosilaga ... deyiladi.
a) ikkinchi (tartibli) hosila b) uchinchi (tartibli) hosila
c) urinma d) differentsial
3. $f''(x)$ bu … .
a) $\lim\limits_{\Delta x \to 0} \frac{f'(x+\Delta x)-f'(x)}{\Delta x}$ b) $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$
c) $\lim\limits_{\Delta x \to 0} \frac{f'(x+\Delta x)+f'(x)}{\Delta x}$ d) $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)+f(x)}{\Delta x}$
4. Ikkinchi (tartibli) hosiladan yana hosila olinsa, u holda ... bo`ladi.
a) uchinchi (tartibli) hosila b) ikkinchi (tartibli) hosila
c) urinma d) differentsial
5. $(n-1)$-tartibli hosiladan olingan hosila (mavjud bo`lgan taqdirda) ... deyiladi.
a) n-tartibli hosila b) $(n-1)$-tartibli hosila
c) $(n+1)$-tartibli hosila d) $(n-2)$-tartibli hosila
6. $y = f(x)$ funksiyaning ikkinchi (tartibli) va unda yuqori (tartibli) hosilalariga ... deyiladi.
a) yuqori tartibli hosilalari b) uchinchi (tartibli) hosilalari
c) ikkinchi (tartibli) hosilalari d) differentsiali
7. $y = f(x)$ funksiyaning ikkinchi tartibli differensialini ko`rsating.
a) $d^2y = f''(x)dx^2$ b) $d^2y = f'(x)dx^2$
c) $d^2y = f''(x)dx^1$ d) $d^2y = f'(x)dx^1$
8. $(n-1)$-tartibli differensialning differensiali … deyiladi.
a) n-tartibli differensial b) $(n-1)$-tartibli differensial
c) 2-tartibli differensial a) 3-tartibli differensial
9. n-tartibli differensial … ko`rinishda belgilanadi.
a) $d^ny = f^{(n)}(x)dx^n$ b) $d^ny = f^{(n-1)}(x)dx^{n-1}$

c) $d^n y = f^{(2)}(x)dx^2$ d) $d^{n-1}y = f^{(n-2)}(x)dx^{n+1}$

10. Agar $y = y(x)$ va $x = x(y)$ funksiyalar o`zaro teskari bo`lsa, y ni teskari funksiyaning ikkinchi tartibli hosilasi toping.

a) $x''_{yy} = -\frac{y''_{xx}}{(y'_x)^3}$ b) $x''_{yy} = 2\frac{y''_{xx}}{(y'_x)^3}$

c) $x''_{yy} = -2\frac{y''_{xx}}{(y'_x)^3}$ d) $x''_{yy} = 3\frac{y''_{xx}}{(y'_x)^3}$

11. Agar $y = y(x)$ va $x = x(y)$ funksiyalar o`zaro teskari bo`lsa, y ni teskari funksiyaning uchinchi tartibli hosilasi toping.

a) $x'''_{yyy} = \frac{3(y''_{xx})^2 - y'''_{xxx}\cdot y'_x}{(y'_x)^5}$ b) $x'''_{yyy} = 2\frac{3(y''_{xx})^2 - y'''_{xxx}\cdot y'_x}{(y'_x)^5}$

c) $x'''_{yyy} = 3\frac{3(y''_{xx})^2 + y'''_{xxx}\cdot y'_x}{(y'_x)^5}$ d) $x'''_{yyy} = \frac{3(y''_{xx})^2 + y'''_{xxx}\cdot y'_x}{(y'_x)^2}$

12. Agar $y = y(x)$ funksiya parametrik $x = \varphi(t)$, $y = \psi(t)$ ko`rinishda berilgan bo`lsa, u holda ikkinchi tartibli hosilasini toping.

a) $y''_{xx} = \frac{(y'_x)'_t}{x'_t}$ b) $y''_{xx} = -2\frac{(y'_x)'_t}{x'_t}$

c) $y''_{xx} = -3\frac{(y'_x)'_t}{x'_t}$ d) $y''_{xx} = 2\frac{(y'_x)'_t}{x'_t}$

13. Agar $y = y(x)$ funksiya parametrik $x = \varphi(t)$, $y = \psi(t)$ ko`rinishda berilgan bo`lsa, u holda ikkinchi tartibli hosilasini toping.

a) $y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t}$ b) $y'''_{xxx} = 2\frac{(y''_{xx})'_t}{x'_t}$

c) $y'''_{xxx} = 3\frac{(y''_{xx})'_t}{x'_t}$ d) $y'''_{xxx} = -3\frac{(y''_{xx})'_t}{x'_t}$

14. $y = f(x)$ funksiya x_0 nuqtadagi Teylor formulasi ko`rsating.

a) $f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1}(x)$

b) $f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x + x_0) + \frac{f''(x_0)}{2!}(x + x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x + x_0)^n + R_{n+1}(x)$

c) $f(x) = f(x_0) - \frac{f'(x_0)}{1!}(x - x_0) - \frac{f''(x_0)}{2!}(x - x_0)^2 - \cdots - \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1}(x)$

d) $f(x) = f(x_0) - \frac{f'(x_0)}{1!}(x + x_0) - \frac{f''(x_0)}{2!}(x + x_0)^2 - \cdots - \frac{f^{(n)}(x_0)}{n!}(x + x_0)^n + R_{n+1}(x)$

15. $y = f(x)$ funksiya x_0 nuqtadagi Makloren formulasi ko`rsating.

a) $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$

b) $f(x) = f(0) - \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 - \cdots + \frac{f^{(n)}(0)}{n!}x^n - \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$

c) $f(x) = f(0) - \frac{f'(0)}{1!}x - \frac{f''(0)}{2!}x^2 - \cdots - \frac{f^{(n)}(0)}{n!}x^n - \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$

d) $f(x) = f(0) - \frac{f'(0)}{1!}x - \frac{f''(0)}{2!}x^2 - \cdots - \frac{f^{(n)}(0)}{n!}x^n$

16. $f(x) = \mathrm{e}^x$ funksiyani $x_0 = 0$ nuqta atrofida yoying.

a) $\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\theta x}$, $\theta \in (0,\ 1)$

b) $\mathrm{e}^x = 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} - \frac{x^{n+1}}{(n+1)!}e^{\theta x}$, $\theta \in (0,\ 1)$

c) $\mathrm{e}^x = 1 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots - \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\theta x}$, $\theta \in (0,\ 1)$

d) $\mathrm{e}^x = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\theta x}$, $\theta \in (0,\ 1)$

17. $f(x) = \sin x$ funksiyani $x_0 = 0$ nuqta atrofida yoying.

a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^n}{n!}\sin\frac{n\pi}{2} + \frac{x^{n+1}}{(n+1)!}\sin\left(\theta x + \frac{(n+1)\pi}{2}\right)$,
$\theta \in (0, 1)$

b) $\sin x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^n}{n!}\sin\frac{n\pi}{2} + \frac{x^{n+1}}{(n+1)!}\sin\left(\theta x + \frac{(n+1)\pi}{2}\right)$,
$\theta \in (0, 1)$

c) $\sin x = x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots - \frac{x^n}{n!}\sin\frac{n\pi}{2} - \frac{x^{n+1}}{(n+1)!}\sin\left(\theta x + \frac{(n+1)\pi}{2}\right)$,
$\theta \in (0, 1)$

d) $\sin x = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots - \frac{x^n}{n!}\sin\frac{n\pi}{2} + \frac{x^{n+1}}{(n+1)!}\sin\left(\theta x + \frac{(n+1)\pi}{2}\right)$,
$\theta \in (0, 1)$

18. $f(x) = \cos x$ funksiyani $x_0 = 0$ nuqta atrofida yoying.

a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{x^n}{n!}\cos\frac{n\pi}{2} + \frac{x^{n+1}}{(n+1)!}\cos\left(\theta x + \frac{(n+1)\pi}{2}\right)$,
$\theta \in (0, 1)$

b) $\cos x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^n}{n!}\cos\frac{n\pi}{2} + \frac{x^{n+1}}{(n+1)!}\cos\left(\theta x + \frac{(n+1)\pi}{2}\right)$,
$\theta \in (0, 1)$

c) $\cos x = -1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{x^6}{6!} - \cdots - \frac{x^n}{n!}\cos\frac{n\pi}{2} - \frac{x^{n+1}}{(n+1)!}\cos\left(\theta x + \frac{(n+1)\pi}{2}\right)$,
$\theta \in (0, 1)$

d) $\cos x = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots - \frac{x^n}{n!}\cos\frac{n\pi}{2} + \frac{x^{n+1}}{(n+1)!}\cos\left(\theta x + \frac{(n+1)\pi}{2}\right)$,

19. $y = x^2 + 3x + 12$ bo`lsa, u holda y'' toping.

a) $y'' = 2$ b) $y'' = -3$ c) $y'' = 2x$ d) $y'' = 0$

20. $y = x^3 + \cos x$ bo`lsa, u holda y''' toping.

a) $y''' = 6 + \sin x$ b) $y''' = 3x^2 + 2\sin x$

c) $y''' = 2 - \sin x$ d) $y''' = 8 - 2\cos x$

21. $y = 4x - 3$ bo`lsa, u holda y'' toping.

a) $y'' = 0$ b) $y'' = 4$ c) $y'' = 4x$ d) $y'' = -4$

22. $y = 3^x$ bo`lsa, u holda y'' toping.

a) $y'' = 3^x\ln^2 3$ b) $y'' = 3x^2$ c) $y'' = \frac{1}{x\ln 3}$ d) $y'' = 3\cos x$

23. $f(x) = \mathrm{e}^x$ funksiyaning $y^{(n)}$ hosilasini toping.

a) $y^{(n)} = \mathrm{e}^x$ b) $y^{(n)} = \mathrm{e}^{-x}$ c) $y^{(n)} = \mathrm{n!}\,\mathrm{e}^x$ d) $y^{(n)} = -\mathrm{n!}\,\mathrm{e}^x$

24. $f(x) = \ln x$ funksiyaning $y^{(n)}$ hosilasini toping.
a) $y^{(n)} = (-1)^{n-1}(n-1)!\,x^{-n}$ b) $y^{(n)} = (-1)^{n-1}(n+2)!\,x^{-n}$
c) $y^{(n)} = (-1)^{n-1}(n-1)!\,x^{n}$ d) $y^{(n)} = (-1)^{n-1}(n+1)!\,x^{-n}$
25. $y = \sin 2x$ bo`lsa, u holda y''' toping.
a) $y''' = -8\cos 2x$ b) $y''' = 8\cos 2x$
c) $y''' = 8\sin 2x$ d) $y''' = -8\sin 2x$
26. $y = x^n$ bo`lsa, u holda $y^{(n)}$ toping.
a) $y^{(n)} = n!$ b) $y^{(n)} = n$ c) $y^{(n)} = -n!$ d) $y^{(n)} = -n$
27. $y = \mathrm{e}^{-x}$ bo`lsa, u holda y''' toping.
a) $y''' = -\mathrm{e}^{-x}$ b) $y''' = \mathrm{e}^{-x}$ c) $y''' = -\mathrm{e}^{x}$ d) $y''' = \mathrm{e}^{x}$
28. $y = x^3 + 2x + 2$ bo`lsa, u holda y'' toping.
a) $y'' = 6x$ b) $y'' = 6x - 2$ c) $y'' = 6$ d) $y'' = -6x$
29. $y = x^3 + 2x + 2$ bo`lsa, u holda $y''(1)$ toping.
a) $y''(1) = 6$ b) $y''(1) = -2$ c) $y''(1) = -4$ d) $y''(1) = -3$
30. $y = \mathrm{e}^{-x}$ bo`lsa, u holda $y'''(0)$ toping.
a) 1 b) -1 c) 0 d) 2

14-§. DIFFERENSIAL HISOBNING ASOSIY TEOREMALARI. ANIQMASLIKLARNI OCHISH

Quyida Differensial hisobning asosiy teoremalari deb ataluvchi Roll teoremasi, Lagranj teoremasi va Koshi teoremasini bayon etamiz. Keyin Differensial hisobning muhim tadbiqlaridan biri – aniqmasliklarni ochishning Lopital qoidasini keltiramiz.

14.1. Roll teoremasi (nazariy qism)

Roll teoremasi. Aytaylik, $y = f(x)$ funksiya $[a,\ b]$ kesmada:

(i) $y = f(x)$ funksiya $[a,\ b]$ kesmada uzluksiz;

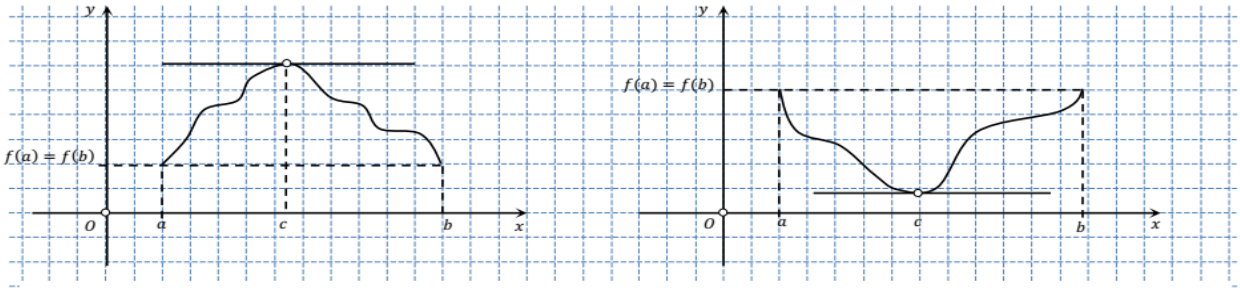
(ii) $y = f(x)$ funksiya $(a,\ b)$ intervalda differensiallanuvchi;

(iii) $f(a) = f(b)$

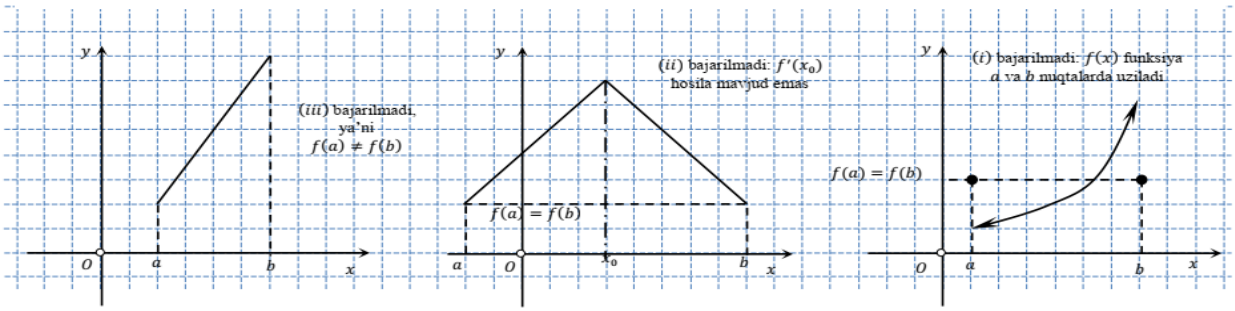
xossalarga ega bo`lsin. U holda shunday $c \in (a,\ b)$ nuqta topiladiki, $f'(c) = 0$ bo`ladi.

Roll teoremasi geometrik nuqtai nazardan $[a,\ b]$ kesmada $(i) - (iii)$ xossalarga ega funksiya uchun $(a,\ b)$ intervalda har doim shunday c nuqta topilib, funksiya grafigining bu nuqtadagi urinmasi Ox o`qiga parallel bo`lishini anglatadi.

Roll teoremasi xulosasida bunday nuqta yagonaligi aytilmagan. Bunday nuqtalar bir nechta bo`lishi mumkin, lekin har doim bunday nuqtadan hech bo`lmaganda bittasi mavjud bo`ladi.



Quyida Roll teoremasining faqatgina bitta shartini bajarmaydigan sodda misollar keltirilgan. Natijada bu funksiyalar uchun teorema xulosasi o`rinli emas, ya'ni barcha $x \in (a,\ b)$ larda $f'(x) \neq 0$.



Roll teoremasidagi 3 ta shartdan bittasi bajarilmasa, teorema xulosasi o`rinli bo`lmay qoladi (rasmga qarang).

Muhim natija. Agar $f(x)$ funksiya $[a,\ b]$ kesmada uzluksiz, $(a,\ b)$ intervalda differensiallanuvchi hamda a, b nuqtalar $f(x) = 0$ tenglamaning ildizlari bo`lsa, u holda bu nuqtalar orasida $f'(x) = 0$ tenglamaning hech bo`lmaganda bitta ildizi mavjud bo`ladi.

14.1.1.-misol. Ushbu funksiyalar Roll teoremasi shartlarini qanoatlantiradimi:

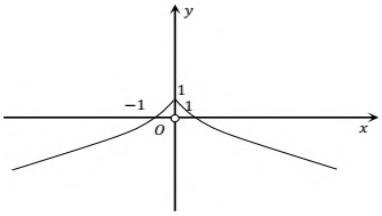
$a)$ $[-1,\ 1]$ kesmada $f(x) = 1 - \sqrt[3]{x^2}$ funksiya?

$b)$ $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ kesmada $f(x) = \ln \sin x$ funksiya?

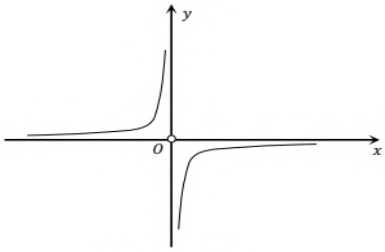
$c)$ $[-1,\ 1]$ kesmada $f(x) = 1 - |x|$ funksiya?

Agar qanoatlantirmasa, nega?

Yechish. $a)$ $f(x) = 1 - \sqrt[3]{x^2}$ funksiya $[-1,\ 1]$ kesmada uzluksiz. Undan tashqari, $f(-1) = f(1) = 0$. Demak, Roll teoremasining (i) va (iii) shartlari bajariladi. $f'(x) = -\frac{2}{3\sqrt[3]{x}}$ hosila $x = 0$ nuqtadan boshqa barcha nuqtalarda mavjud. Bu nuqta $[-1,\ 1]$ kesmaning ichki nuqtasi, ya'ni $0 \in (-1,\ 1)$ bo`lgani uchun Roll teoremasining $(ii)$ sharti bajarilmaydi. Bundan bu funksiya uchun Roll teoremasi xulosasini, $f'(c) = 0$ bo`ladigan $c \in (-1,\ 1)$ mavjudligini tasdiqlash mumkin emasligi kelib chiqadi. Haqiqatan ham, $f'(x) = 0$, yoki shuning o`zi $\frac{2}{3\sqrt[3]{x}} = 0$ tenglama yechimga ega emas, jumladan, $x \in (-1,\ 1)$ da $f'(x) \neq 0$.



$f(x) = 1 - \sqrt[3]{x^2}$ funksiya grafigi



funksiya hosilasi $f'(x) = -\frac{2}{3\sqrt[3]{x}}$ ning grafigi

$b)$ $f(x) = \ln \sin x$ funksiya $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ kesmada uzluksiz ((i) bajarildi). Hisoblashlar $f\left(\frac{\pi}{6}\right) = f\left(\frac{5\pi}{6}\right) = \ln\frac{1}{2}$ ekanligini ko`rsatadi ($(iii)$ bajarildi). $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ intervalda $f'(x) = \operatorname{ctg} x$ hosila mavjud ($(ii)$ bajarildi). Demak, $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ intervalda $f'(x) = 0$, ya'ni $\operatorname{ctg} x = 0$ tenglamaning hech bo`lmaganda bitta ildizi mavjud. Bu ildiz $x = \frac{\pi}{2}$ bo`ladi. Demak, bu funksiya uchun Roll teoremasi xulosasi $f'\left(\frac{\pi}{2}\right) = 0$ tenglikdan iborat ekan.

$c)$ $f(x) = 1 - |x|$ funksiya $[-1,\ 1]$ kesmada uzluksiz ((i) bajarildi), bunda $f(-1) = f(1) = 0$ ((iii) bajarildi). Lekin $f(x) = 1 - |x|$ funksiyaning

$$f'(x) = (1 - |x|)' = \begin{cases} 1, & x < 0, \\ \text{mavjud emas} & x = 0, \\ -1, & x > 0 \end{cases}$$

hosilasi $[-1,\ 1]$ kesmaning ichki $x = 0$ nuqtasida mavjud emas. Shuning uchun Roll teoremasini bu funksiyaga qo`llab bo`lmaydi.

14.2. Lagranj teoremasi

Lagranj teoremasi. $y = f(x)$ funksiya $[a,\ b]$ kesmada:

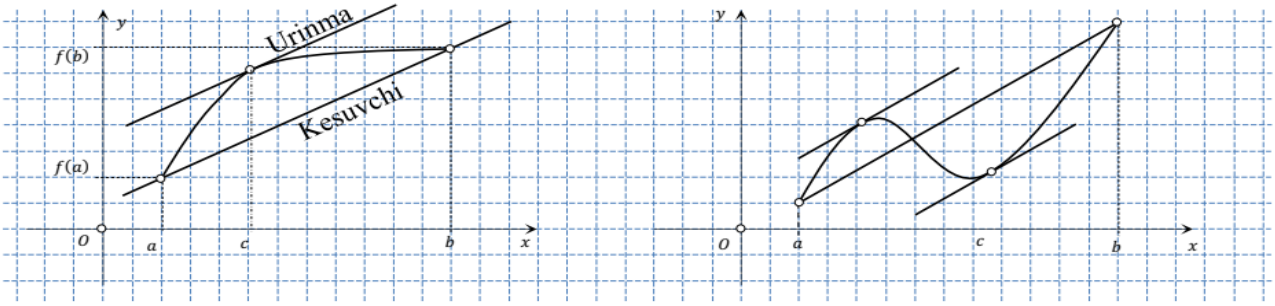
(i) $y = f(x)$ funksiya $[a,\ b]$ kesmada uzluksiz;

(ii) $y = f(x)$ funksiya $(a,\ b)$ intervalda differensiallanuvchi

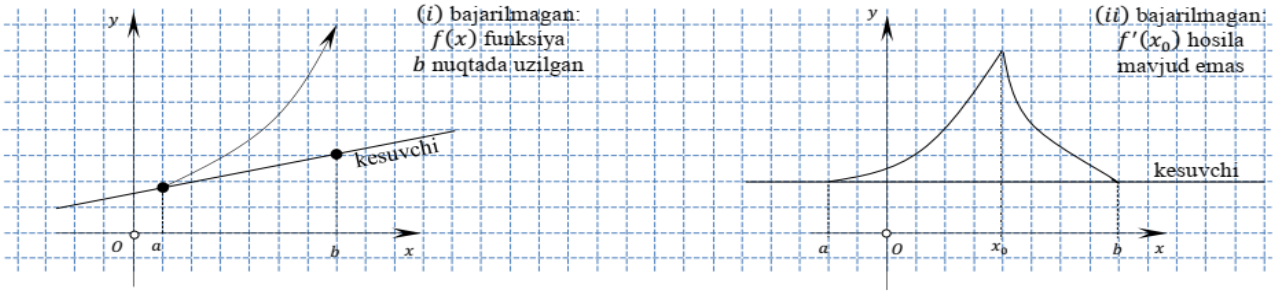
xossalarga ega bo`lsin. U holda shunday $c \in (a,\ b)$ nuqta topiladiki, $f'(c) = \frac{f(b)-f(a)}{b-a}$ bo`ladi.

Lagranj teoremasi geometrik nuqtai nazardan $[a,\ b]$ kesmada $(i) - (ii)$ xossalarga ega funksiya uchun $(a,\ b)$ intervalda har doim shunday c nuqta topilib, funksiya grafigining bu nuqtadagi urinmasi grafikning oxirlaridan o`tuvchi kesuvchiga parallel bo`ladi.

Lagranj teoremasi xulosasida kesuvchiga parallel bunday urinma yagonaligi aytilmagan. Bunday urinmalar bir nechta bo`lishi mumkin, lekin har doim bunday urinmadan hech bo`lmaganda bittasi mavjud bo`ladi.



Quyida Lagranj teoremasi shartlaridan birini bajarmaydigan sodda funksiyalar misollari keltirilgan. Natijada hosila mavjud bo`lgan har bir $x \in (a,\ b)$ nuqtadagi urinma grafik oxirlaridan o`tuvchi kesuvchiga parallel bo`lmaydi.



Agar Lagranj teoremasida qo`shimcha $f(a) = f(b)$ shart talab etilsa, ravshanki, $f'(c) = 0$ bo`ladi (Roll teoremasi kelib chiqadi). Demak, Roll teoremasi Lagranj teoremasining xususiy holi ekan.

14.2.1.-misol. Lagranj teoremasidan foydalanib, $\ln(1 + \mathrm{e})$ ifodaning qiymatini baholang.

Yechish. $y = \ln x$ funksiya $[\mathrm{e},\ 1 + \mathrm{e}]$ kesmada uzluksiz va $[\mathrm{e},\ 1 + \mathrm{e}]$ ning barcha ichki nuqtalarida chekli hosilaga ega. Demak, $y = \ln x$ funksiya Lagranj teoremasi shartlarini qanoatlantiradi. Hosilani topamiz: $f'(x) = \frac{1}{x}$. Bu hosila uchun Lagranj teoremasi xulosasini qo`llaymiz: shunday $c \in (\mathrm{e},\ 1 + \mathrm{e})$ nuqta mavjudki,

$$\frac{1}{c} = \frac{\ln(1+\mathrm{e}) - \ln \mathrm{e}}{1+\mathrm{e}-\mathrm{e}}$$

bo`ladi. Bu yerdan $c = \frac{1}{\ln(1+\mathrm{e})-1}$. Mavjudligiga ko`ra $\mathrm{e} < c < 1 + \mathrm{e}$ ekanligidan quyidagilarga ega bo`lamiz

$$e < \frac{1}{\ln(1+e)-1} < 1+e \Rightarrow \frac{1}{1+e} < \ln(1+e)-1 < \frac{1}{e} \Rightarrow \frac{2+e}{1+e} < \ln(1+e) < \frac{1+e}{e}.$$

Shunday qilib, $\ln(1+e)$ ifoda uchun $1{,}26 < \ln(1+e) < 1{,}37$ bahoga ega bo`ldik. Uning taqribiy qiymati $\ln(1+e) \approx 1{,}31$.

14.3. Koshi teoremasi

Koshi teoremasi. $[a,\ b]$ kesmada berilgan $f(x)$ va $g(x)$ funksiyalar:

(i) $f(x)$ va $g(x)$ funksiyalar $[a,\ b]$ kesmada uzluksiz;

(ii) $f(x)$ va $g(x)$ funksiyalar $(a,\ b)$ intervalda differensiallanuvchi hamda bu intervalda $g'(x) \neq 0$

shartlarni qanoatlantirsin. U holda shunday $c \in (a,\ b)$ nuqta topiladiki, bu nuqtada ushbu tenglik o`rinli bo`ladi

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}\ .$$

Koshi teoremasining geometrik ma'nosi $(a,\ b)$ intervalda $f'(c)$ va $g'(c)$ hosilalar nisbati funksiyalarning$f(b)-f(a)$ va $g(b)-g(a)$ orttirmalari nisbatiga teng bo`lasigan c nuqta topilishidan iborat.

Shuni ta'kidlaymizki, $g(b)-g(a) \neq 0$ bo`ladi. Aks holda $g(a) = g(b)$ bo`lib, Roll teoremasidan $(a,\ b)$ dan shunday c nuqta topilib, $g'(c) = 0$ bo`lishi kelib chiqadi. Bu esa Koshi teoremasidagi $(a,\ b)$ da $g'(x) \neq 0$ bo`lishi shartiga ziddir.

Agar $g(x) = x$ bo`lsa, $f'(c) = \frac{f(b)-f(a)}{b-a}$ tenglikka (ya'ni, Lagranj teoremasiga) ega bo`lamiz. Demak, Lagranj teoremasi Koshi teoremasining xususiy holi ekan.

14.3.1.-misol. $f(x) = x^2 - 2x + 3$ va $g(x) = x^3 - 7x^2 + 20x - 5$ funksiyalar $[1,\ 4]$ kesmada Koshi teoremasi shartlarini bajarishini ko`rsating va mos c nuqtani toping.

Yechish. $f(x) = x^2 - 2x + 3$ va $g(x) = x^3 - 7x^2 + 20x - 5$ funksiyalar hamma yerda (ya'ni $(-\infty, +\infty)$ da), xususan, $[1,\ 4]$ kesmada uzluksiz;

ularning $f'(x) = 2x - 2$ va $g'(x) = 3x^2 - 14x + 20$ hosilalari $[1,\ 4]$ da chekli; undan tashqari x ning barcha haqiqiy qiymatlarida $g'(x)$ hosila nolga aylanmaydi (chunki discriminant manfiy). Shunday qilib, Koshi formulasini berilgan bu funksiyalarga tadbiq etish mumkin, ya'ni shunday $c \in (1,\ 4)$ nuqta topilib,

$$\frac{f'(c)}{g'(c)} = \frac{f(4)-f(1)}{g(4)-g(1)} = \frac{4^2-2\cdot4+3-(1^2-2\cdot1+3)}{4^3-7\cdot4^2+20\cdot4-5-(1^3-7\cdot1^2+20\cdot1-5)} = \frac{11-2}{27-9} = \frac{9}{18} = \frac{1}{2}$$

bo`ladi. Bu yerdan

$$g'(c) = 2f'(c),$$
$$3c^2 - 14c + 20 = 2(2c - 2\ ),$$
$$3c^2 - 18c + 24 = 0,$$
$$c^2 - 6c + 8 = 0,$$
$$c_1 = 2, \qquad c_2 = 4.$$

Bu ikkita qiymatdan faqatgina $c_1 = 2$ nuqta $[1,\ 4]$ kesmaning ichki nuqtasi bo`ladi. Demak, izlangan qiymat $c = c_1 = 2$.

14.4. Differensial hisobning aniqmasliklarni ochishda qo`llanilishi

Differensial hisob turlicha aniqmasliklarni ochishda unumli ishlatilishi mumkinligini ta'kidlash joiz.

14.4.1. $\frac{0}{0}$ ko`rinishidagi aniqmaslikni ochish

$f(x)$ va $g(x)$ funksiyalar:

(i) $f(x)$ va $g(x)$ funksiyalar a nuqtaning biror atrofidagi (balki a nuqtadan boshqa) nuqtalarida hosilalarga ega hamda $g'(x) \neq 0$;

(ii) $\lim\limits_{x\to a} f(x) = 0$ va $\lim\limits_{x\to a} g(x) = 0$;

(iii) (chekli yoki cheksiz) $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ limit mavjud

shartlarni qanoatlantirsin. U holda $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ limit mavjud hamda

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

tenglik o`rinli.

Oxirgi tenglik *Lopital qoidasi* deyiladi va uni quyidagi shaklda bayon etish mumkin:

Cheksiz kichik funksiyalar nisbatining $x \to a$ dagi limitini bu funksiyalar hosilalari nisbatining $x \to a$ dagi limiti bilan almashtirish mumkin.

Ayrim hollarda Lopital qoidasini qo`llagach, yana $\frac{0}{0}$ tipidagi aniqmaslik hosil bo`lshi mumkin. Bu holda olingan natijaga yana Lopital qoidasi qo`llaniladi. Shunday qilib, aniqmasliklarni ochish Lopital qoidasini qayta qo`llashni talab etishi mumkin.

Lopital qoidasi bir tomonlama limitlarda ham qo`llanilishi mumkin, ya'ni ushbu tengliklar o`rinli

$$\lim_{x\to a-0} \frac{f(x)}{g(x)} = \lim_{x\to a-0} \frac{f'(x)}{g'(x)} \quad \text{va} \quad \lim_{x\to a+0} \frac{f(x)}{g(x)} = \lim_{x\to a+0} \frac{f'(x)}{g'(x)}.$$

Undan tashqari, Lopital qoidasi $x \to \infty$ bo`lganda ham o`rinli:

$$\lim_{x\to -\infty} \frac{f(x)}{g(x)} = \lim_{x\to -\infty} \frac{f'(x)}{g'(x)} \quad \text{va} \quad \lim_{x\to +\infty} \frac{f(x)}{g(x)} = \lim_{x\to +\infty} \frac{f'(x)}{g'(x)}.$$

14.4.2. $\frac{\infty}{\infty}$ ko`rinishidagi aniqmaslikni ochish

Agar berilgan funksiyalar cheksiz katta funksiyalar bo`lsa, Lopital qoidasini tashkil etuvchi quyidagi jumla o`rinli bo`ladi.

$f(x)$ va $g(x)$ funksiyalar:

(i) $f(x)$ va $g(x)$ funksiyalar a nuqtaning biror atrofidagi (balki a nuqtadan boshqa) nuqtalarida hosilalarga ega hamda $g'(x) \neq 0$;

(ii) $\lim\limits_{x\to a} f(x) = \infty$ va $\lim\limits_{x\to a} g(x) = \infty$;

(iii) (chekli yoki cheksiz) $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ limit mavjud

shartlarni qanoatlantirsin. U holda $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ limit mavjud hamda

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

tenglik o`rinli.

Bu holda ham ushbu tengliklar o`rinli

$$\lim_{x\to a-0}\frac{f(x)}{g(x)}=\lim_{x\to a-0}\frac{f'(x)}{g'(x)} \quad \text{va} \quad \lim_{x\to a+0}\frac{f(x)}{g(x)}=\lim_{x\to a+0}\frac{f'(x)}{g'(x)};$$

$$\lim_{x\to -\infty}\frac{f(x)}{g(x)}=\lim_{x\to -\infty}\frac{f'(x)}{g'(x)} \quad \text{va} \quad \lim_{x\to +\infty}\frac{f(x)}{g(x)}=\lim_{x\to +\infty}\frac{f'(x)}{g'(x)}.$$

$0\cdot\infty$ yoki $\infty-\infty$ ko`rinishdagi aniqmasliklar $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko`rinishdagi aniqmasliklarga keltirish mumkin.

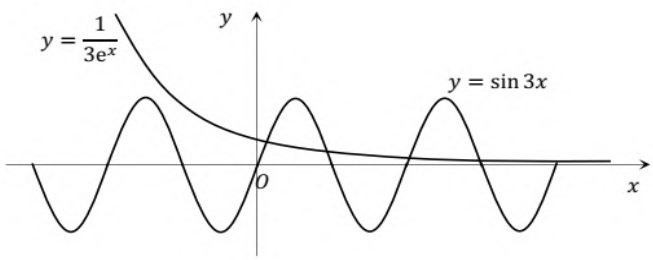
1^∞, 0^0, ∞^0 ko`rinishdagi aniqmasliklar quyidagicha ochiladi. Aytaylik, $f(x)$ va $g(x)$ funksiyalar uchun $\lim\limits_{x\to a}f(x)=1$ va $\lim\limits_{x\to a}g(x)=\infty$. U holda $\lim\limits_{x\to a}\big(f(x)\big)^{(g(x))}$ limit $1^\infty$ ko`rinishdagi aniqmaslikni beradi. $y=\big(f(x)\big)^{g(x)}$ belgilash kiritib, uni logarifmlaymiz:

$$\ln y=\ln\big(f(x)\big)^{g(x)}=g(x)\ln f(x)=\frac{\ln f(x)}{\frac{1}{g(x)}}.$$

Natijada $\lim\limits_{x\to a}\frac{\ln f(x)}{\frac{1}{g(x)}}$ limit $\frac{0}{0}$ aniqmaslikni beradi. Unga Lopital qoidasini qo`llash mumkin. Bu qoida qo`llangach, $\lim\limits_{x\to a}\frac{\ln f(x)}{\frac{1}{g(x)}}=k$ ekanligi ma'lum bo`lgan bo`lsin. Demak, $\lim\limits_{x\to a}\ln y=k$. Logarifmik funksiya, ya'ni ln uzluksizligidan $\lim\limits_{x\to a}\ln y=\ln\lim\limits_{x\to a}y=k$ bo`lib , undan $\lim\limits_{x\to a}y=\mathrm{e}^k$, ya'ni $\ln\big(f(x)\big)^{g(x)}=\mathrm{e}^k$.

14.4.2.1.-misol. Lopital qoidasini qo`llab, $x\to 0$ da $y=\frac{\ln(1+x^2)}{\cos 3x-\mathrm{e}^{-x}}$ funksiyaning limitini toping.

Yechish. $\lim\limits_{x\to 0}\ln(1+x^2)=0$ va и $\lim\limits_{x\to 0}(\cos 3x-\mathrm{e}^{-x})=0$ tengliklar o`rinli. Shuning uchun $\lim\limits_{x\to 0}\frac{\ln(1+x^2)}{\cos 3x-\mathrm{e}^{-x}}$ limit $\frac{0}{0}$ tipidagi aniqmaslik bo`ladi. Nol $(x=0)$ ning har qanday atrofida $(\ln(1+x^2))'=\frac{2x}{1+x^2}$ va $(\cos 3x-\mathrm{e}^{-x})'=-3\sin 3x+\mathrm{e}^{-x}$ hosilalar mavjud. Chizmadan



$$\sin 3x=\frac{1}{3\mathrm{e}^x}$$

tenglama cheksiz ko`p yechimga ega ekanligi ko`rinib turibdi. p bu tenglamaning (moduli bo`yicha) eng kichik yechim bo`lsin. U holda nolning $(-p,\ p)$ atrofida $-3\sin 3x+\mathrm{e}^{-x}\neq 0$. Undan tashqari, $\lim\limits_{x\to 0}(\ln(1+x^2))'=\lim\limits_{x\to 0}\frac{2x}{1+x^2}=0$ va

$\lim\limits_{x\to 0}(\cos 3x - e^{-x})' = \lim\limits_{x\to 0}(-3\sin 3x + e^{-x}) = 1$. Shuning uchun Lopital qoidasini qo`llash mumkin:

$$\lim_{x\to 0}\frac{\ln(1+x^2)}{\cos 3x - e^{-x}} = \lim_{x\to 0}\frac{(\ln(1+x^2))'}{(\cos 3x - e^{-x})'} = \lim_{x\to 0}\frac{\frac{2x}{1+x^2}}{-3\sin 3x + e^{-x}} = \frac{0}{1} = 0\ .$$

14.4.2.2.-misol. $\lim\limits_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$ limitni toping.

Yechish. Berilgan bu limit $x \to 1+0$ da $\infty - \infty$ ko`rinishidagi aniqmaslikni beradi. Uni avvalo $\frac{0}{0}$ ko`rinishidagi aniqmaslikka keltiriladi, keyin Lopital qoidasi qo`llaniladi:

$$\lim_{x\to 1+0}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = (\infty - \infty) = \lim_{x\to 1+0}\frac{x - 1 - \ln x}{(x-1)\ln x} = \left(\frac{0}{0}\right) =$$

$$= \lim_{x\to 1+0}\frac{1 - \frac{1}{x}}{\ln x + 1 - \frac{1}{x}} = \lim_{x\to 1+0}\frac{x-1}{x\ln x + x - 1} = \left(\frac{0}{0}\right) = \lim_{x\to 1+0}\frac{1}{\ln x + 2} = \frac{1}{2}.$$

$x > 0$ bo`lib, $x \to 1-0$ da berilgan limitning qiymati quyidagiga teng:

$$\lim_{x\to 1-0}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = +\infty + \infty = +\infty.$$

14.4.2.3.-misol. $\lim\limits_{x\to 0} x^n \ln x$ limitni toping $(n > 0)$.

Yechish. Limit ostidagi funksiyaning aniqlanish sohasi $(0, +\infty)$ bo`lgani uchun $x \to 0$ holatni $x \to 0+$ deb tushunish talab etiladi.

$$\lim_{x\to 0} x^n \ln x = (0\cdot\infty) = \lim_{x\to 0}\frac{\ln x}{x^{-n}} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to 0}\frac{\frac{1}{x}}{-nx^{-n-1}} = -\frac{1}{n}\lim_{x\to 0} x^n = 0.$$

14.4.2.4.-misol. $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}$ limitni toping.

Yechish. Logarifmning aniqlanish sohasini e'tiborga olgan holda ushbularga ega bo`lamiz:

$$\lim_{x\to 0+}\left(\frac{1}{x}\right)^{\sin x} = (\infty^0) = e^{\ln \lim\limits_{x\to 0+}\left(\frac{1}{x}\right)^{\sin x}} = e^{\lim\limits_{x\to 0+}\ln\left(\frac{1}{x}\right)^{\sin x}} = e^{\lim\limits_{x\to 0+}\left(\sin x \ln\left(\frac{1}{x}\right)\right)} =$$

$$= e^{\lim\limits_{x\to 0+}\frac{\ln\left(\frac{1}{x}\right)}{\frac{1}{\sin x}}} = \left(e^{\frac{\infty}{\infty}}\right) = e^{\lim\limits_{x\to 0+}\frac{\left(\ln\left(\frac{1}{x}\right)\right)'}{\left(\frac{1}{\sin x}\right)'}} = e^{\lim\limits_{x\to 0+}\frac{-\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}}} = e^{\lim\limits_{x\to 0+}\frac{\sin^2 x}{x\cos x}} = \left(e^{\frac{0}{0}}\right) =$$

$$= e^{\lim\limits_{x\to 0+}\frac{(\sin^2 x)'}{(x\cos x)'}} = e^{\lim\limits_{x\to 0+}\frac{2\sin x\cos x}{\cos x - x\sin x}} = e^{\frac{0}{1}} = e^0 = 1.$$

$$\lim_{x\to 0-}\left(\frac{1}{x}\right)^{\sin x} = (\infty^0) = e^{\ln \lim\limits_{x\to 0-}\left(-\frac{1}{x}\right)^{\sin x}} = e^{\lim\limits_{x\to 0-}\ln\left(-\frac{1}{x}\right)^{\sin x}} = e^{\lim\limits_{x\to 0+}\left(\sin x \ln\left(-\frac{1}{x}\right)\right)} =$$

$$= e^{\lim\limits_{x\to 0+}\frac{\ln\left(-\frac{1}{x}\right)}{\frac{1}{\sin x}}} = \left(e^{\frac{\infty}{\infty}}\right) = e^{\lim\limits_{x\to 0+}\frac{\left(\ln\left(-\frac{1}{x}\right)\right)'}{\left(\frac{1}{\sin x}\right)'}} = e^{\lim\limits_{x\to 0+}\frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}}} = e^{-\lim\limits_{x\to 0+}\frac{\sin^2 x}{x\cos x}} = \left(e^{\frac{0}{0}}\right) =$$

$$= e^{-\lim\limits_{x\to 0+}\frac{(\sin^2 x)'}{(x\cos x)'}} = e^{-\lim\limits_{x\to 0+}\frac{2\sin x\cos x}{\cos x - x\sin x}} = e^{-\frac{0}{1}} = e^0 = 1.$$

Demak, $\lim\limits_{x\to 0+}\left(\frac{1}{x}\right)^{\sin x} = \lim\limits_{x\to 0-}\left(\frac{1}{x}\right)^{\sin x} = 1$. Bundan $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\sin x} = 1$.

Mustahkamlash uchun mashqlar

Differensial hisobning asosiy teoremalariga oid masalalar

1. Lopital qoidasidan foydalanib limtni toping.

1. $\lim\limits_{x\to 0}\frac{tg\,3x}{\sin 2x}$
2. $\lim\limits_{x\to 0}\frac{1-\cos 6x}{x\sin 3x}$
3. $\lim\limits_{x\to 1}\frac{\sin(2(x-1))}{x^2-7x+6}$
4. $\lim\limits_{x\to\infty}\frac{\ln(x+3)}{\sqrt{x+3}}$
5. . $\lim\limits_{x\to 0}\frac{tg\,x-x}{x-\sin x}$
6. $\lim\limits_{x\to 0}\frac{1-\cos x^2}{x^2-\sin x^2}$
7. $\lim\limits_{x\to\infty}\frac{\ln(x+5)}{\sqrt[4]{x+3}}$
8. $\lim\limits_{x\to 0}\frac{a^{\ln x}-x}{x-1}$
9. $\lim\limits_{x\to 1}\frac{1-4\sin^2\frac{\pi x}{6}}{1-x^2}$
10. $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)$
11. $\lim\limits_{x\to 2}\frac{\sin 3(x-2)}{x^2-3x+2}$
12. $\lim\limits_{x\to 0}\frac{e^{5x}-1}{\sin 10x}$
13. $\lim\limits_{x\to 0}\frac{\ln(1+7x)}{\sin 7x}$
14. $\lim\limits_{x\to 0}\frac{tg\,8x}{\sin 3x}$

2. Quyidagi funksiyalarni Makloren formulasi bo‘yicha yozing:

1. $y=\ln x$
2. $y=\frac{1}{x}$
3. $y=\cos x$
4. $y=\sin x$
5. $y=\frac{1}{x+5}$
6. $y=e^{-2x}$

O`z-o`zini tekshirish uchun savollar

1. Roll teoremasini tushuntiring.
2. Lagranj teoremasi tushuntiring.
3. Koshi teoremasi tushuntiring.
4. Differensial hisobning aniqmasliklarni ochishda qo`llanilishini tushuntiring.

O`z-o`zini tekshirish uchun testlar

Differensial hisobning asosiy teoremalari. Aniqmasliklarni ochishga oid testlar

1. $y=f(x)$ funksiya $[a,\ b]$ kesmada: (i) $y=f(x)$ funksiya $[a,\ b]$ kesmada uzluksiz; (ii) $y=f(x)$ funksiya $(a,\ b)$ intervalda differensiallanuvchi; (iii) $f(a)=f(b)$ xossalarga ega bo`lsin. U holda shunday $c\in(a,\ b)$ nuqta topiladiki, $f'(c)=0$ bo`ladi. Bu kimning teoremasi?

a) Roll teoremasi b) Lagranj teoremasi

c) Koshi teoremasi d) Lopital qoidasi

2. $y = f(x)$ funksiya $[a,\ b]$ kesmada: (i) $y = f(x)$ funksiya $[a,\ b]$ kesmada uzluksiz; (ii) $y = f(x)$ funksiya $(a,\ b)$ intervalda differensiallanuvchi xossalarga ega bo`lsin. U holda shunday $c \in (a,\ b)$ nuqta topiladiki, $f'(c) = \frac{f(b)-f(a)}{b-a}$ bo`ladi. Bu kimning teoremasi?

a) Roll teoremasi b) Lagranj teoremasi
c) Koshi teoremasi d) Lopital qoidasi

3. $[a,\ b]$ kesmada berilgan $f(x)$ va $g(x)$ funksiyalar: (i) $f(x)$ va $g(x)$ funksiyalar $[a,\ b]$ kesmada uzluksiz; (ii) $f(x)$ va $g(x)$ funksiyalar $(a,\ b)$ intervalda differensiallanuvchi hamda bu intervalda $g'(x) \neq 0$ shartlarni qanoatlantirsin. U holda shunday $c \in (a,\ b)$ nuqta topiladiki, bu nuqtada $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$ tenglik o`rinli bo`ladi. Bu kimning teoremasi?

a) Roll teoremasi b) Lagranj teoremasi
c) Koshi teoremasi d) Lopital qoidasi

4. $f(x)$ va $g(x)$ funksiyalar: (i) $f(x)$ va $g(x)$ funksiyalar a nuqtaning biror atrofidagi (balki a nuqtadan boshqa) nuqtalarida hosilalarga ega hamda $g'(x) \neq 0$; (ii) $\lim\limits_{x\to a} f(x) = 0$ va $\lim\limits_{x\to a} g(x) = 0$; (iii) (chekli yoki cheksiz) $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ limit mavjud shartlarni qanoatlantirsin. U holda $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ limit mavjud hamda $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ tenglik o`rinli. Bu kimning teoremasi?

a) Roll teoremasi b) Lagranj teoremasi
c) Koshi teoremasi d) Lopital qoidasi

5. $f(x)$ va $g(x)$ funksiyalar: (i) $f(x)$ va $g(x)$ funksiyalar a nuqtaning biror atrofidagi (balki a nuqtadan boshqa) nuqtalarida hosilalarga ega hamda $g'(x) \neq 0$; (ii) $\lim\limits_{x\to a} f(x) = \infty$ va $\lim\limits_{x\to a} g(x) = \infty$; (iii) (chekli yoki cheksiz) $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ limit mavjud shartlarni qanoatlantirsin. U holda $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ limit mavjud hamda $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ tenglik o`rinli. Bu kimning teoremasi?

a) Roll teoremasi b) Lagranj teoremasi
c) Koshi teoremasi d) Lopital qoidasi

6. $0 \cdot \infty$ yoki $\infty - \infty$ ko`rinishdagi aniqmasliklarga Lopital qoidasi qanday qo`llaniladi?

a) $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko`rinishdagi aniqmasliklarga keltirish orqali
b) 1^{∞} ko`rinishdagi aniqmasliklarga keltirish orqali
c) 0^0 ko`rinishdagi aniqmasliklarga keltirish orqali
d) ∞^0 ko`rinishdagi aniqmasliklarga keltirish orqali

7. Δx bu …

a) argument orttirmasi b) funksiya orttirmasi
c) funksiya grafigi d) funksiya differentsiali

8. Δy bu …

a) funksiya orttirmasi b) argument orttirmasi

c) funksiya grafigi d) funksiya differentsiali

9. $\Delta y =$

a) $f(x_0 + \Delta x) - f(x_0)$ b) $x - x_0$

c) kx d) $tgx - tg\ x_0$

10. $\Delta x =$

a) $f(x_0 + \Delta x) - f(x_0)$ b) $x - x_0$

c) kx d) $tgx - tg\ x_0$

11. y' … bildiradi.

a) $y = f(x)$ funksiyaning hosilasini

b) $y = f(x)$ funksiyaning boshlang`ichini

c) $y = f(x)$ funksiyaning differentsialini

d) $y = f(x)$ funksiyaning orttirmasini

12. $y = f'(x_0)(x - x_0) + f(x_0)$ bu…

a) $f(x)$ funksiyaga x_0 nuqtada o`tkazilgan urinma tenglamasi.

b) $f(x)$ funksiyaga x_0 nuqtada o`tkazilgan narmal tenglamasi.

c) $s = f(x)$ qoida bilan harakatlanayotgan jismning x_0 nuqtadagi tezlanishi.

d) $f(x_0)$ va $f'(x_0)$ lar orasidagi bog`lanish.

13. $y = -\frac{1}{f'(x_0)}(x - x_0) + f(x_0)$

a) $f(x)$ funksiyaga x_0 nuqtada o`tkazilgan urinma tenglamasi.

b) $f(x)$ funksiyaga x_0 nuqtada o`tkazilgan narmal tenglamasi.

c) $s = f(x)$ qoida bilan harakatlanayotgan jismning x_0 nuqtadagi tezlanishi.

d) $f(x_0)$ va $f'(x_0)$ lar orasidagi bog`lanish.

14. $dy =$

a) $f'(x)dx$ b) $f(x)dx$ c) kx d) $tgx - tg\ x_0$

15. $dx =$

a) $x - x_0$ b) $f(x_0 + \Delta x) - f(x_0)$ c) kx d) $tgx - tg\ x_0$

16. $(f \pm g)' =$

a) $f' \pm g'$ b) $f'g + fg'$ c) $f' \pm g'$ d) $\frac{f'g - fg'}{g'^2}$

17. $(f \cdot g)' =$

a) $f' \pm g'$ b) $f'g + fg'$ c) $f' \pm g'$ d) $\frac{f'g - fg'}{g'^2}$

18. $\left(\frac{f}{g}\right)' =$

a) $f' \pm g'$ b) $f'g + fg'$ c) $f' \pm g'$ d) $\frac{f'g - fg'}{g'^2}$

19. $\sin' 2x =$

a) $2\cos(2x)$ b) $-2\cos(2x)$ c) $2\sin(2x)$ d) $-2\sin(2x)$

20. $\cos'(\pi x) =$

a) $-\pi\sin(\pi x)$ b) $\pi\sin(\pi x)$ c) $-\sin(\pi x)$ a) $\sin(\pi x)$

21. Agar $\lim\limits_{x\to a} f(x) = 0$ va $\lim\limits_{x\to a} g(x) = 0$ bo`lsa, u holda $\lim\limits_{x\to a}\frac{f(x)}{g(x)} =$ limit … ko`rinishdagi aniqmaslik deyiladi.

a) $\frac{0}{0}$ b) $\frac{\infty}{\infty}$ c) 1^{∞} d) 0^0

22. Agar $\lim\limits_{x\to a} f(x) = 0$ va $\lim\limits_{x\to a} g(x) = \infty$ bo`lsa, u holda $\lim\limits_{x\to a} f(x)g(x)$ limit … ko`rinishdagi aniqmaslik deyiladi.

a) $0 \cdot \infty$ b) $\frac{\infty}{\infty}$ c) 1^{∞} d) 0^0

23. $f(x) = x^2 + \frac{1}{x}$, $f'^{(x)} = ?$

a) 11 b) 0 c) -1 d) 6

24. Agar $y = \cos^2 x$ bo`lsa, y''=?

a) $-2\cos x \cdot \sin x$ b) $-\cos 2x$

c) $2\cos x \cdot \sin x$ d) $\sin 2x$

25. Agar $y = \sin(x+1)$ bo`lsa, y'=?

a) $\cos(x+1)$ b) $\cos x + 1$

c) $\sin x + x$ d) $\sin x + 1$

26. $\lim\limits_{x\to 0} \frac{\operatorname{tg} 3x}{\sin 2x} = ?$

a) $\frac{3}{2}$ b) $\frac{3}{5}$ c) 3 d) 2

27. $\lim\limits_{x\to 0} \frac{1-\cos 6x}{x \sin 3x} = ?$

a) 6 b) 5 c) 3 d) 2

28. $\lim\limits_{x\to 1} \frac{\sin(2(x-1))}{x^2-7x+6} = ?$

a) $-\frac{2}{5}$ b) $\frac{3}{5}$ c) 3 d) 2

29. $\lim\limits_{x\to\infty} \left(\frac{3x+2}{3x-1}\right)^{4x-1} = ?$

a) e^x b) $\frac{3}{5}$ c) 3 d) 2

30. $\lim\limits_{x\to\infty} ((2x+1)(\ln(3x+1) - \ln(3x-2))) = ?$

a) 2 b) 7 c) 3 d) 2

15-§. HOSILANING TADBIQLARI

15.1. Funksiyaning doimiylik va monotonlik shartlari (nazariy qism)

Funksiyaning **doimiylik alomatini** bayon etamiz: agar $y = f(x)$ funksiya differensiallanuvchi bo`lib, biror $[a,\ b]$ oraliqning barcha nuqtalarida $f'(x) = 0$ bo`lsa, u holda $f(x)$ funksiya o`z qiymatini bu oraliqda doimo o`zgartirmay saqlaydi.

$(a,\ b)$ intervalda doimiy $f(x)$ funksiyaning hosilasi $f'(x) = 0$ bo`lishi va uning grafigiga $(a,\ b)$ intervaldagi har bir nuqtada o`tkazilgan urinmasi Ox o`qiga parallel bo`lishi funksiyaning doimiylik alomatining **geometrik ma'nosi**dir.

15.1.1.-misol. Funksiyaning doimiylik alomatidan foydalanib, quyidagi formulalarni ketirib chiqaring:

$a)$ $\arcsin x + \arccos x = \frac{\pi}{2}$;

$b)$ $\sin^2 x = \frac{1-\cos 2x}{2}$;

$c)$ $\arccos\frac{1-x^2}{1+x^2} = 2\operatorname{arctg} x$, bunda $0 \le x < \infty$.

Yechish. $a)$ $[-1,\ 1]$ kesmada aniqlangan ushbu funksiyani kiritamiz

$$f(x) = \arcsin x + \arccos x.$$

Bu funksiyaning mazkur kesma ichidagi hosilasi nolga teng:

$$f'(x) = (\arcsin x + \arccos x)' = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0, \quad -1 < x < 1.$$

Funksiyaning doimiylik alomatiga asosan

$$\arcsin x + \arccos x = C, \qquad -1 < x < 1.$$

bo`ladi. $C$ o`zgarmasni aniqlash uchun, masalan, $x = 0$ deb, $C = \frac{\pi}{2}$ ni topamiz. Bundan

$$\arcsin x + \arccos x = \frac{\pi}{2}, \qquad -1 < x < 1.$$

$x = -1$ bo`lsa,

$$\arcsin(-1) + \arccos(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2},$$

$x = 1$ bo`lsa,

$$\arcsin 1 + \arccos 1 = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

ekanligidan $-1 \le x \le 1$ da ushbu tenglikka erishamiz

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

$b)$ Son o`qi $-\infty < x < +\infty$ da aniqlangan ushbu funksiyani kiritamiz

$$f(x) = \sin^2 x + \frac{1}{2}\cos 2x\ .$$

keyin $f(x) = \frac{1}{2}$ ekanligini ko`rsatamiz.

Bu funksiya hosilasi har doim nolga teng:

$$f'(x) = \left(\sin^2 x + \frac{1}{2}\cos 2x\right)' = 2\sin x\cos x - \sin 2x = 0.$$

Funksiyaning doimiylik alomatiga asosan

$$\sin^2 x + \frac{1}{2}\cos 2x = C\ , \quad -\infty < x < +\infty.$$

C o`zgarmasni aniqlash uchun, masalan, $x = 0$ deb, $C = \frac{1}{2}$ ni topamiz. Bundan

$$\sin^2 x + \frac{1}{2}\cos 2x = \frac{1}{2},$$

yoki

$$\sin^2 x = \frac{1-\cos 2x}{2}.$$

$c)$ $0 \le x < \infty$ da aniqlangan ushbu funksiyani kiritamiz

$$f(x) = \arccos\frac{1-x^2}{1+x^2} - 2\operatorname{arctg} x\ .$$

Barcha $x > 0$ ushbu hosilaga egamiz:

$$f'(x) = -\frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{-4x}{(1+x^2)^2} - \frac{2}{1+x^2} = \frac{1}{2x} \cdot \frac{4x}{1+x^2} - \frac{2}{1+x^2} = 0.$$

Funksiyaning doimiylik alomatiga asosan $x > 0$ da ushbuga egamiz

$$\arccos\frac{1-x^2}{1+x^2} - 2\operatorname{arctg} x = C\ .$$

C o`zgarmasni aniqlash uchun, masalan $x = 1$ deb olamiz. U holda:

$$C = \arccos 0 - 2\operatorname{arctg} 1 = \frac{\pi}{2} - 2 \cdot \frac{\pi}{4} = 0.$$

$x = 0$ bo`lsa,

$$C = \arccos 1 - 2arctg\ 0 = 0 - 0 = 0.$$

Shunday qilib, $0 \le x < \infty$ da, $\arccos\frac{1-x^2}{1+x^2} - 2\operatorname{arctg} x = 0$, ya'ni

$$\arccos\frac{1-x^2}{1+x^2} = 2\operatorname{arctg} x\ .$$

Funksiyaning monotonlik shartlari quyidagi tasdiqlarda o`z aksini topgan:

1) Agar $f(x)$ funksiya $[a,\ b]$ kesmada differensiallanuvchi va o`suvchi bo`lsa, u holda $[a,\ b]$ kesmada $f'(x) \ge 0$ bo`ladi.

Agar $f(x)$ funksiya $[a,\ b]$ kesmada uzluksiz, $(a,\ b)$ intervalda differensiallanuvchi va $f'(x) > 0$ bo`lsa, u holda $f(x)$ funksiya $[a,\ b]$ kesmada o`suvchi bo`ladi.

2) Agar $f(x)$ funksiya $[a,\ b]$ kesmada differensiallanuvchi va kamayuvchi bo`lsa, u holda $[a,\ b]$ kesmada $f'(x) \le 0$ bo`ladi.

Agar $f(x)$ funksiya $[a,\ b]$ kesmada uzluksiz, $(a,\ b)$ intervalda differensiallanuvchi va $f'(x) < 0$ bo`lsa, u holda $f(x)$ funksiya $[a,\ b]$ kesmada kamayuvchi bo`ladi.

Monotonlik shartlarining geometrik ma'nosi quyidagidan iborat: $(a,\ b)$ intervalda o`suvchi funksiyaning grafigiga o`tkazilgan har qanday urinma Ox o`qning musbat yo`nalishi bilan o`tkir burchak hosil qiladi; $(a,\ b)$ intervalda kamayuvchi funksiyaning grafigiga o`tkazilgan har qanday urinma Ox o`qning musbat yo`nalishi bilan o`tmas burchak hosil qiladi.

15.1.2.-misol. $y = x^4$ funksiyaning monotonlik sohalarini ko`rsating.

Yechish. Hosilani topamiz:

$$y' = 4x^3.$$

$y' < 0$, ya'ni $4x^3 < 0$ tengsizlikni yechib, funksiyaning kamayish sohasini topamiz: $x < 0$.

Shunga o`xshash, funksiyaning o`sish sohasi topiladi: $x > 0$.

15.2. Funksiyaning ekstremumlari

Agar:

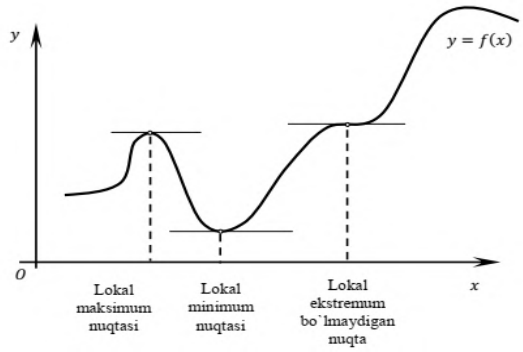
1) $f(x)$ funksiya x_0 nuqtada va uning qandaydir atrofida aniqlangan;
2) shu atrofning x_0 dan farqli barcha x nuqtalarida $f(x) < f(x_0)$

shartlar bajarilsa, x_0 nuqta $f(x)$ funksiyaning *lokal maksimumi* deyiladi.

Agar:

1) $f(x)$ funksiya x_0 nuqtada va uning qandaydir atrofida aniqlangan;
2) shu atrofning x_0 dan farqli barcha x nuqtalarida $f(x) > f(x_0)$

shartlar bajarilsa, x_0 nuqta $f(x)$ funksiyaning *lokal minimumi* deyiladi.



Lokal maksimum va lokal minimum iboralari uchun umumiy nom *lokal ekstremum* iborasi ishlatiladi.

15.3. Ekstremum mavjud bo`lishining zaruriy va yetarli shartlari

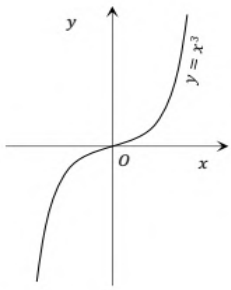
15.3.1. Ekstremum mavjud bo`lishining zaruriy sharti

Ekstremum mavjud bo`lishining **zaruriy sharti** ushbu tasdiqdan iborat: Agar $f(x)$ funksiya $x_0$ nuqtada differensiallanuvchi bo`lib, bu nuqta funsiyaning lokal ekstremumi bo`lsa, u holda $f'(x_0) = 0$ bo`ladi.

Lekin x_0 nuqtada differensiallanuvchi $f(x)$ funksiya uchun $f'(x_0) = 0$ ekanligidan x_0 nuqta $f(x)$ funksiyaning ekstremum nuqtasi bo`lishi kelib chiqmaydi.

15.3.1.1.-Misol. $x = 0$ qiymat $y = x^3$ funksiyaning lokal ekstremum nuqtasi bo`ladimi?

Yechish. Hosilani topamiz: $y' = 3x^2$, va uni nolga tenglashtiramiz: $3x^2 = 0$. Bundan $x = 0$. Ravshanki, $y(0) = 0$. Har qanday $x < 0$ uchun $x^3 < 0$, natijada $y(x) < y(0)$ bo`ladi. Shunga o`xshash barcha $x > 0$ uchun $y(x) > y(0)$ bo`ladi. Shunday qilib, $y'(0) = 0$ bo`lishiga qaramasdan $x = 0$ qiymat $y = x^3$ funksiyaning lokal ekstremumi emas.



Quyidagi misollarni o`rganamiz.

15.3.1.2.-misol. $y = |x|$ funksiyaning lokal ekstremumlarini ko`rsating.

Yechish. Berilgan funksiya $x = 0$ nuqtada differensiallanmaydi. Ammo 0 dan farqli barcha x larda $y(x) = |x| > 0$ bo`ladi. Demak, $x = 0$ – funksiyaning lokal minimum nuqtasi bo`ladi.

15.3.1.3.-misol. $y = \sqrt[3]{x}$ funksiya hosilasining uzilish nuqtalari funksiyaning ekstremum nuqtalari bo`ladimi?

Yechish. Hosila $y' = \frac{1}{3\sqrt[3]{x^2}}$ bo`lib, u $x = 0$ nuqrada uziladi. Biroq bu nuqta funksiyaning ekstremum nuqtasi bo`lmaydi: $x < 0$ bo`lganda $\sqrt[3]{x} < 0$, ya'ni $y(x) < y(0)$, shuningdek, $x > 0$ bo`lganda $y(x) > y(0)$ bo`ladi.

Shunday qilib, funksiya faqatgina ikkita holda ekstremumga ega bo`lishi mumkin: hosila mavjud va bu hosila nolga teng nuqtalarda, yoki hosila mavjud bo`lmagan nuqtalarda.

Funksiyaning mavjud hosilasi nolga teng yoki hosilasi mavjud bo`lmagan nuqtalar shu funksiyaning *kritik nuqtalari* deyiladi.

15.3.2. Lokal ekstremum mavjudligining yetarli sharti

Aytaylik, $f(x)$ funksiya kritik x_0 nuqtani saqlovchi biror oraliqda uzluksiz, shu oraliqning (balki x_0 nuqtadan tashqari) barcha nuqtalarida differensiallanuvchi bo`lsin.

Lokal ekstremum mavjudligining **yetarli sharti** ushbu tasdiqdan iborat: Agar son o`qida $x_0$ nuqtaning chap tomonidan o`ng tomoniga o`tishda $f'(x)$ hosila ishorasini:

plyusdan minusga o`zgartirsa, $x_0$ nuqta $f(x)$ ning lokal maksimum nuqtasi bo`ladi;

minusdan plyusga o`zgartirsa, $x_0$ nuqta $f(x)$ ning lokal minimum nuqtasi bo`ladi;

o`zgartimasa, $x_0$ nuqta $f(x)$ ning lokal ekstremum nuqtasi bo`lmaydi.

Shunday qilib, $f(x)$ funksiyaning ekstremumlarini topishda quyidagi algoritmdan foydalanish mumkin:

1. Funksiyaning kritik nuqtalarini topish.
2. Bu kritik nuqtalardan faqatgina funksiya aniqlanish sohasining ichki nuqtalari bo`ladiganlarini tanlab olish.
3. Son o`qida tanlangan har bir kritik nuqtaning chap tomonidan o`ng tomoniga o`tishda $f'(x)$ hosila ishorasi o`zgarishini o`rganish.
4. Lokal ekstremum mavjudligining yetarli shartidan foydalanib, funksiyaning ekstremum nuqtalarini ko`rsatish hamda funksiyaning bu nuqtalardagi qiymatlarini hisoblash.

15.3.2.1.-Misol. $y = \frac{x^3}{3} - 2x^2 + 3x + 1$ funksiya ekstremumlarini izlang.

Yechish. $y' = x^2 - 4x + 3$ hosilani olamiz. $x^2 - 4x + 3 = 0$ tenglamani yechib, uning $x_1 = 1$, $x_2 = 3$ ildizlarini topamiz. $y' = x^2 - 4x + 3$ hosila har doim uzluksiz. Demak, bu ikkita ildizdan boshqa kritik nuqtalar mavjud emas. Ushbu jadvalni tuzamiz

x	$(-\infty, 1)$	1	$(1, 3)$	3	$(3, +\infty)$
y'	$+$	0	$-$	0	$+$
y	↗	max	↘	min	↗

Shunday qilib,

$$y_{\max} = y(1) = 2\frac{1}{3}, \qquad y_{\min} = y(3) = 1.$$

Amaliy masalalarni yechishda lokal ekstremum mavjudligining quyidagi yetarli shartidan foydalanish qulay bo`ladi:

$f(x)$ funksiya x_0 nuqta atrofida ikki marta differensiallanuvchi va $f'(x_0) = 0$ bo`lsin. Agar:

$f''(x_0) > 0$ bo`lsa, x_0 nuqta $f(x)$ funksiyaning lokal minimum nuqtasi;

$f''(x_0) < 0$ bo`lsa, $x_0$ nuqta $f(x)$ funksiyaning lokal maksimum nuqtasi bo`ladi.

15.4. Funksiyaning eng katta va eng kichik qiymatlari

Aytaylik, $f(x)$ funksiya $[a,\ b]$ kesmada aniqlangan va $x_0 \in [a,\ b]$ bo`lsin. Agarda barcha $x \in [a,\ b]$ uchun:

$f(x) \leq f(x_0)$ tengsizlik o`rinli bo`lsa, $f(x)$ funksiya x_0 nuqtada *eng katta qiymatga* erishadi;

$f(x) \geq f(x_0)$ tengsizlik o`rinli bo`lsa, $f(x)$ funksiya x_0 nuqtada *eng kichik qiymatga* erishadi

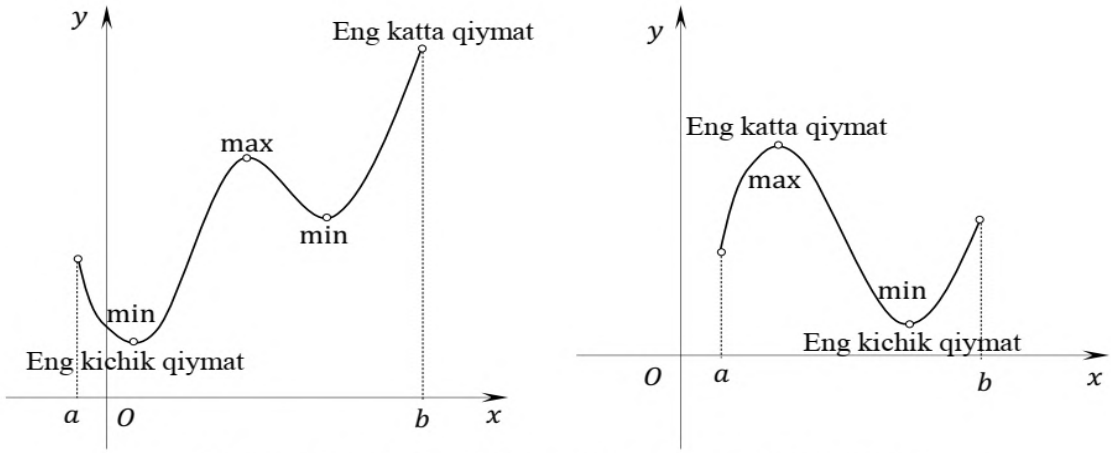
deyiladi.

Agar $f(x)$ funksiya $[a,\ b]$ kesmada **uzluksiz** bo`lsa, u holda bu funksiya shu kesmada o`zining eng katta va eng kichik qiymatlariga erishadi.

$f(x)$ funksiyaning $[a,\ b]$ kesmadagi eng katta qiymatini topish uchun:

1. funksiyaning $[a,\ b]$ kesmadagi barcha maksimum nuqtalaridagi qiymatlari topiladi;
2. $f(a)$ va $f(b)$ qiymatlar hisoblanadi;
3. hosil bo`lgan qiymatlarning eng kattasi tanlanadi; bu qiymat funksiyaning kesmadagi eng katta qiymati ifodalaydi.

Funksiyaning kesmadagi eng kichik qiymatini topishda ham xuddi shunga o`xshash yo`l tutiladi.



Maksimum eng katta qiymat, minimum eng kichik qiymat bo`lishi mumkin.
Eng katta qiymat maksimum bo`lishi, eng kichik qiymat minimum bo`lishi shart emas.

15.4.1.-misol. $y = x^3 - 3x + 3$ funksiyaning $\left[-3, \frac{3}{2}\right]$ kesmadagi eng katta va eng kichik qiymatlarini toping.

Yechish. $y' = 3x^2 - 3$ hosilani topib, uni nolga tenglaymiz:

$$3x^2 - 3 = 0\ .$$

Hosil bo`lgan tenglamani yechib, kritik nuqtalarni topamiz: $x_1 = -1$, $x_2 = 1$. Ushbu jadvalni tuzamiz

x	$(-3,-1)$	-1	$(-1,1)$	1	$\left(1,\frac{3}{2}\right)$
y'	$+$	0	$-$	0	$+$
y	$\nearrow$	max	$\searrow$	min	$\nearrow$

Bu yerdan

$$y_{max} = y(-1) = 5, \qquad y_{min} = y(1) = 1.$$

Endi funksiyaning $\left[-3,\frac{3}{2}\right]$ kesma oxirlaridagi qiymatlarini hisoblaymiz:

$$y(-3) = -15, \qquad y\left(\frac{3}{2}\right) = \frac{15}{8}.$$

Shunday qilib, $y = x^3 - 3x + 3$ funksiyaning $\left[-3,\frac{3}{2}\right]$ kesmadagi eng katta qiymati:

$$y(-1) = 5,$$

eng kichik qiymati:

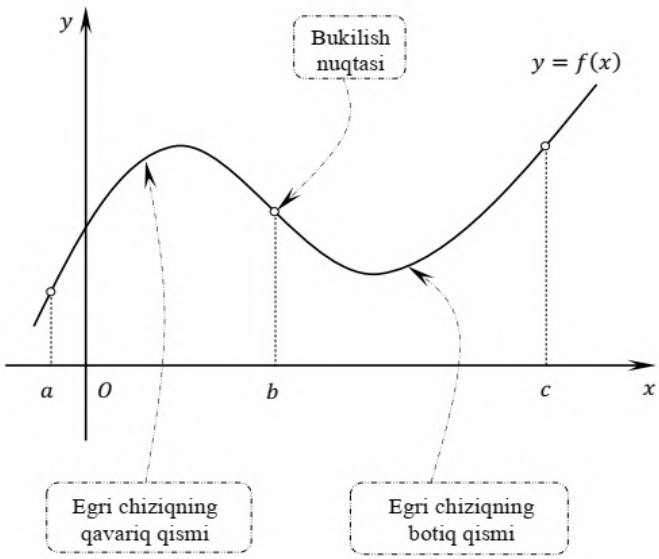
$$y(-3) = -15.$$

15.5. Funksiya grafigining qavariqligi, botiqligi, bukilish nuqtasi

Funksiyani o`rganishda ko`pincha uning grafigi qaysi tomonga botganlini aniqlash talab etiladi.

Differensiallanuvchi $y = f(x)$ funksiyaning grafigi unga $(a,\ b)$ intervaldan olingan har bir nuqtada o`tkazilgan urinmadan pastda joylashgan bo`lsa, bu grafik shu intervalda *qavariq* (yoki *yuqoriga qavariq*) deyiladi.

Differensiallanuvchi $y = f(x)$ funksiyaning grafigi unga $(b,\ c)$ intervaldan olingan har bir nuqtada o`tkazilgan urinmadan tepada joylashgan bo`lsa, bu grafik shu intervalda *botiq* (yoki *pastga qavariq*) deyiladi.



Uzluksiz egri chiziqning qavariq qismini uning botiq qismidan ajratuvchi nuqta egri chiziqning *bukilish nuqtasi* deyiladi.

$y = f(x)$ funksiya grafigining biror oraliqda qavariqligi (botiqligi) **ning yetarli shartlarini** keltiramiz:

Agar $(a,\ b)$ intervalning barcha nuqtalarida $f''(x) < 0$ bo`lsa, u holda bu intervalda $y = f(x)$ funksiya grafigi qavariq bo`ladi;

Agar $(a,\ b)$ intervalning barcha nuqtalarida $f''(x) > 0$ bo`lsa, u holda bu intervalda $y = f(x)$ funksiya grafigi botiq bo`ladi.

Bukilish nuqta bo`lish sharti.

Agar $f''(x)$ ikkinchi hosila biror x_0 nuqtada nolga teng yoki mavjud bo`lmasa, hamda son o`qida x_0 nuqtaning bir tarafidan boshqa tarafiga o`tganda bu ikkinchi hosila ishorasini o`zgartirsa, u holda x_0 nuqta $y = f(x)$ funksiya grafigining bukilish nuqtasi bo`ladi. Be yerda $x_0$ nuqtada $y = f(x)$ funksiya grafigiga o`tkazilgan urinma mavjud deb qaralyapti.

15.5.1.-misol. Gauss egri chizig`i $y = \mathrm{e}^{-x^2}$ ning bukilsih nuqtalari hamda qavariqlik va botiqlik intervallarini toping.

Yechish. Birinchi va ikkinchi hosilalarni topamiz:

$$y' = -2x\mathrm{e}^{-x^2},$$
$$y'' = 2(2x^2 - 1)\mathrm{e}^{-x^2}.$$

Bu hosilalar har doim aniqlangan.

$y' = 0$ tenglamaning $x = 0$ yechimini topamiz. Demak, $x = 0$ kritik nuqta ekan.

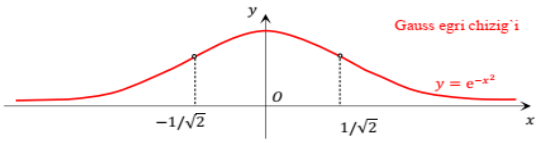
$y'' = 0$ bo`ladiga x ning qiymatlarini topamiz:

$$2x^2 - 1 = 0,$$
$$x_1 = -\frac{1}{\sqrt{2}}, \qquad x_2 = \frac{1}{\sqrt{2}}.$$

Ushbu jadvalni tuzamiz

x	$\left(-\infty, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}}$	$\left(-\frac{1}{\sqrt{2}}, 0\right)$	0	$\left(0, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}, +\infty\right)$
y'	+	+	+	0	−	−	−
y''	+	0	−	−1	−	0	+
y	↗ ∪	Bukilish nuqtasi	↗ ∩	max	↘ ∩	Bukilish nuqtasi	↘ ∪

Bunda: ↗ simbol funksiya o`sishini, ↘ simbol funksiya kamayishin, ∪ egri chiziq botiqligini, ∩ egri chiziq qavariqligini anglatadi.



Shunday qilib, bukilish nuqtalari: $-\frac{1}{\sqrt{2}}$ va $\frac{1}{\sqrt{2}}$,

Botiqlik sohasi: $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, +\infty\right)$,

Qavariqlik sohasi: $\left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)$.

15.6. Funksiya grafigining asimptotalari

$y = f(x)$ funksiya grafigi – $f(x)$ egri chiziq va l to`g`ri chiziq berilgan bo`lsin. $f(x)$ egri chiziqdagi $M$ nuqta koordinata boshidan egri chiziq bo`ylab cheksiz uzoqlashganda M nuqtadan l to`g`ri chiziqqacha masofa nolga intilsa, l to`g`ri chiziq bu $f(x)$ egri chiziqning asimptotasi deyiladi.

Agar

$$\lim_{x\to a} f(x) = \infty, \quad \lim_{x\to a-0} f(x) = \infty, \quad \lim_{x\to a+0} f(x) = \infty$$

tengliklardan birontasi bajarilsa, u holda $x = a$ to`g`ri chiziq $y = f(x)$ egri chiziqning *vertikal asimptotasi* deyiladi.

Vertikal asimptotalar odatda ikkinchi tur uzilish nuqtalaridan o`tadi.

Agar $\lim\limits_{x\to\infty} f(x) = k$ bo`lsa, u holda $y = f(x)$ egri chiziq $y = k$ *gorizontal asimptota* ega deyiladi

$y = f(x)$ egri chiziqning og`ma asimptotasi odatda $y = kx + b$ ko`rinishda izlanadi. Uning mavjudligi quyidagi tasdiqda ifodalangan.

$y = kx + b$ to`g`ri chiziq $y = f(x)$ egri chiziqning asimptotasi bo`lishi uchun $\lim\limits_{x\to\infty}(f(x) - kx - b) = 0$ tenglik bajarilishi zarur va yetarli.

Amaliy masalalarni yechishda bu koeffisiyentlar quyidagicha topiladi:

$$k = \lim_{x\to\infty} \frac{f(x)}{x}, \qquad b = \lim_{x\to\infty}(f(x) - kx).$$

Bu limitlardan bittasi mavjud bo`lmasa yoki cheksizga teng bo`lsa, u holda egri chiziq og`ma asimptotaga ega bo`lmaydi.

Gorizontal asimptotani og`ma asimptotaning $k = 0$ bo`lgandagi xususiy holi deb qarash mumkin.

15.6.1.-misol. $y = \frac{x^2+2x-1}{x}$ egri chiziqning asimptotalarini toping.

Yechish. $x = 0$ nuqta funksiyaning uzilish nuqtasi bo`lgani uchun shu nuqtada vertikal asimptotalarni izlaymiz:

$$\lim_{x\to 0-} \frac{x^2+2x-1}{x} = \lim_{x\to 0-}\left(x + 2 - \frac{1}{x}\right) = +\infty\,,$$

$$\lim_{x\to 0+} \frac{x^2+2x-1}{x} = \lim_{x\to 0+}\left(x + 2 - \frac{1}{x}\right) = -\infty\,.$$

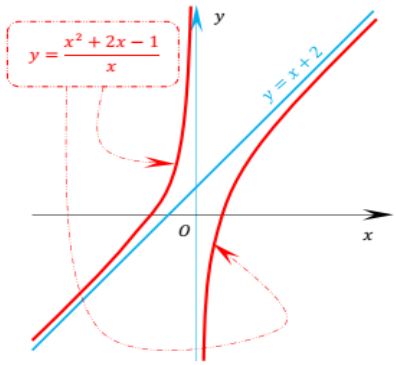
Demak, berilgan egri chiziq uchun $x = 0$ vertikal asimptotaga ega ekan.

$\lim\limits_{x\to\pm\infty} \frac{x^2+2x-1}{x} = \lim\limits_{x\to\pm\infty}\left(x + 2 - \frac{1}{x}\right) = \pm\infty$ bo`lgani uchun qaralayotgan egri chiziq gorizontal asimptotaga ega emas.

O`rganilayotgan egri chiziqning o`g`ma asimptotalarini izlaymiz:

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2+2x-1}{x} = \lim_{x\to\infty}\left(1 + \frac{2}{x} - \frac{1}{x^2}\right) = 1\,,$$

$$b = \lim_{x\to\infty}(f(x) - kx) = \lim_{x\to\infty}\left(\frac{x^2+2x-1}{x} - x\right) = \lim_{x\to\infty}\left(\frac{2x-1}{x}\right) = 2\,.$$



Natijada $y = x + 2$ to`g`ri chiziq berilgan egri chiziqning asimptotasi bo`ladi.

Egri chiziq va asimptotaning tekislikda o`zaro joylashuvini aniqlashtirish uchun x aynan bitta qiymatida egri chiziq va asimptota ordinatalarining ayirmasini qaraymiz:

$$\frac{x^2+2x-1}{x} - (x + 2) = -\frac{1}{x}.$$

$x > 0$ bo`lganda bu ayirma manfiy, demak bu holda egri chiziq asimptotadan pastda yotadi;

$x < 0$ bo`lganda bu ayirma musbat, demak, bu holda egri chiziq asimptotadan tepada yotadi.

15.7. Funksiyani tekshirish va uning grafigini chizishning umumiy sxemasi

“Funksiyani tekshirish” deganda odatda quyidagilarni izlash tushuniladi:

1) funksiyaning aniqlanish sohasi;
2) funksiya grafigining koordinata o`qlari bilan kesishish nuqtalari;
3) funksiyaning uzilish nuqtalari;
4) funksiyaning monotonlik intervallari;
5) funksiyaning ekstremum nuqtalari, eng katta va eng kichik qiymatlari;
6) funksiya grafigining qavariqlik va botiqlik sohalari, bukilish nuqtalari;
7) funksiya grafigining asimptotalari.

Olib borilgan tekshirishlar asosida funksiya grafigi (grafik eskizi) chiziladi.

Bu ishlar bajarilgach grafikni chizish tushunarli bo`lmasa, funksiyani toq yoki juft ekanligi, davriyligi, grafikning bir necha nuqtalarini belgilash va funksiyaning boshqa xususiyatlarini qo`shimcha o`rganish mumkin.

15.7.1.-misol. $y = \frac{x}{1+x^2}$ funksiyani tekshiring va grafigini chizing.

Yechish. 1) $D = (-\infty, +\infty)$ – funksiyaning aniqlanish sohasi; $x > 0$ bo`lganda $y > 0$, va $x < 0$ bo`lganda $y < 0$ ekanligini ta’kidlaymiz.

2) $x = 0$ da $y = 0$ bo`ladi. Grafik $Oy$ o`qni $O(0,\ 0)$ nuqtada kesadi. Agarda $y = 0$ bo`lsa, $x = 0$ bo`ladi. Demak, grafik Ox o`qni $O(0,\ 0)$ nuqtada kesadi.

3) Funksiya har doim uzluksiz, ya’ni uzilish nuqtasi yo`q.

4) Monotonlik intervallarini, ekstremum nuqtalarini, funksiyaning bu nuqtalardagi qiymatlarini topamiz.

Buning uchun $y' = \frac{1-x^2}{(1+x^2)^2}$ hosilani topib, nolga tenglab

$$\frac{1-x^2}{(1+x^2)^2} = 0,$$

kritik nuqtalarini topamiz:

$$x_1 = -1, \qquad x_2 = 1.$$

Ushbu jadvalni tuzamiz

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, +\infty)$
y'	$-$	0	$+$	0	$-$
y	↘	min	↗	max	↘

Shunday qilib,

kamayish sohasi $(-\infty, -1) \cup (1, +\infty)$;

o`sish sohasi $(-1, 1)$;

minimum nuqtasi $x = -1$, $\quad y_{min} = y(-1) = -\frac{1}{2}$;

maksimum nuqtasi $x = 1$, $\quad y_{max} = y(1) = \frac{1}{2}$.

5) Qavariqlik, botiqlik sohalarini va bukilish nuqtalarini izlaymiz. Buning uchun ikkinchi hosilani topib

$$y'' = \frac{2x(x^2-3)}{(1+x^2)^3},$$

uni nolga tenglab

$$\frac{2x(x^2-3)}{(1+x^2)^3}=0,$$

ushbu qiymatlarni topamiz

$$x_1=-\sqrt{3}, \qquad x_2=0, \qquad x_3=\sqrt{3}.$$

Ushbu jadvalni tuzamiz

x	$(-\infty,-\sqrt{3}$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},+\infty)$
y''	$-$	0	$+$	0	$-$	0	$+$
y	$\cap$	Bukilish nuqtasi	$\cup$	Bukilish nuqtasi	$\cap$	Bukilish nuqtasi	$\cup$

Demak,

Qavariqlik sohasi $(-\infty,\ -\sqrt{3})\cup(0,\ \sqrt{3})$;

Botiqlik sohasi $(-\sqrt{3},\ 0)\cup(\sqrt{3},+\infty)$;

Bukilish nuqtalari $x_1=-\sqrt{3}$, $x_2=0$, $x_3=\sqrt{3}$.

6) Funksiya grafigi asimptotalarini topamiz.

Berilgan egri chiziq vertikal asimptotaga ega emas, chunki funksiya $-\frac{1}{2}$ va $\frac{1}{2}$ sonlari bilan chegaralangan.

$\lim\limits_{x\to\pm\infty}\frac{x}{1+x^2}=0$ ekanligidan $y=0$ to`g`ri chiziq egri chiziqning gorizontal asimptotasi bo`ladi.

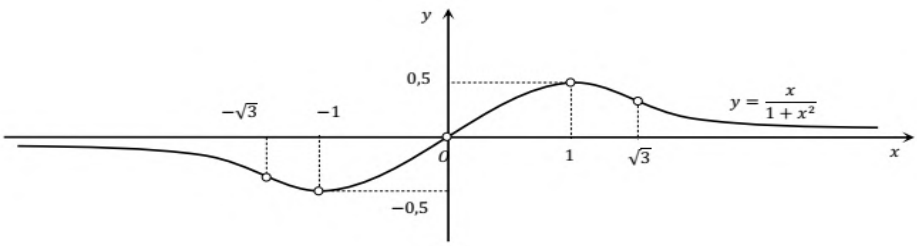
O`g`ma asimptotalarni izlaymiz:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{x}{x(1+x^2)}=0\ ,$$

$$b=\lim_{x\to\pm\infty}(f(x)-kx)=\lim_{x\to\pm\infty}\frac{x}{1+x^2}=0\ .$$

Yana $y=0$ tenglamaga ega bo`ldik. Shunday qilib, $y=0$ – yagona asimptota ekan.

Grafikni chizamiz.



Mustahkamlash uchun mashqlar

Hosilaning tadbiqlariga oid masalalar

1. Quyidagi funksiyalarni to`la tekshiring va grafigini chizing.

1. $y=\frac{x^2-2x+2}{x-1}$ 2. $y=\frac{x+1}{(x-1)^2}$

3. $y=e^{\frac{1}{5+x}}$ 4. $y=\frac{x}{9-x}$

5. $y=\frac{-x^2+4x-4}{x}$ 6. $y=\frac{x^2}{4x^2-1}$

7. $y = \frac{\ln x}{\sqrt{x}}$ 8. $y = x + \frac{\ln x}{x}$

9. $y = x - \ln(1 + x^2)$ 10. $y = \frac{x^3}{x^2 - x + 1}$

2. $y = f(x)$ funksiyaning $[a, b]$ kesmadagi eng katta va eng kichik qiymatlarini toping.

1. $y = \ln(x^2 - 2x + 2)$, $[0,3]$
2. $y = \frac{3x}{x^2+1}$, $[0,5]$
3. $y = \frac{2x-1}{(x-2)^2}$, $\left[-\frac{1}{2}, 0\right]$
4. $y = (x + 2)e^{1-x}$, $[-2,2]$
5. $y = \ln(x^2 - 2x + 4)$, $\left[-1, \frac{3}{2}\right]$
6. $y = \frac{x^3}{x^2-2x+2}$, $[-1,1]$
7. $y = \left(\frac{x+1}{x^3}\right)^3$, $[1,2]$
8. $y = \frac{(x^4+4)}{x^2}$, $[1,2]$

O`z-o`zini tekshirish uchun savollar

1. Funksiyaning doimiylik va monotonlik shartlarini ayting.
2. Funksiyaning ekstremumlari deb nimaga aytiladi va qanday topiladi?
3. Funksiyaning eng katta va eng kichik qiymatlari qanday topiladi?
4. Funksiya grafigining qavariqligi, botiqligi, bukilish nuqtasi qanday topiladi?
5. Funksiya grafigining asimptotalari deb nimaga aytiladi va qanday topiladi?
6. Funksiyani tekshirish va uning grafigini chizishning umumiy sxemasini ayting.

O`z-o`zini tekshirish uchun testlar

Hosilaning tadbiqlariga oid testlar

1. Agar $y = f(x)$ funksiya differensiallanuvchi bo`lib, biror $[a,\ b]$ oraliqning barcha nuqtalarida $f'(x) = 0$ bo`lsa, u holda bu oraliqda ... ,o`ladi.

a) $f(x) = C$ b) $f(x) = x$ c) $f(x) = -x$ d) $f(x) = x^2$

2. Agar $f(x)$ funksiya $[a,\ b]$ kesmada differensiallanuvchi va o`suvchi bo`lsa, u holda $[a,\ b]$ kesmada ... bo`ladi.

a) $f'(x) \geq 0$ b) $f'(x) \leq 0$ c) $f'(x) \geq 1$ b) $f'(x) \leq 1$

3. Agar $f(x)$ funksiya $[a,\ b]$ kesmada uzluksiz, $(a,\ b)$ intervalda differensiallanuvchi va $f'(x) > 0$ bo`lsa, u holda $f(x)$ funksiya $[a,\ b]$ kesmada ... bo`ladi.

a) o`suvchi b) kamayuvchi c) o`zgarmas d) nolga teng

4. Agar $f(x)$ funksiya $[a,\ b]$ kesmada differensiallanuvchi va kamayuvchi bo`lsa, u holda $[a,\ b]$ kesmada ... bo`ladi.

a) $f'(x) \geq 0$ b) $f'(x) \leq 0$ c) $f'(x) \geq 1$ b) $f'(x) \leq 1$

5. Agar $f(x)$ funksiya $[a, b]$ kesmada uzluksiz, (a, b) intervalda differensiallanuvchi va $f'(x) < 0$ bo`lsa, u holda $f(x)$ funksiya $[a, b]$ kesmada ... bo`ladi.

a) o`suvchi b) kamayuvchi c) o`zgarmas d) nolga teng

6. Agar: 1) $f(x)$ funksiya x_0 nuqtada va uning qandaydir atrofida aniqlangan; 2) shu atrofning x_0 dan farqli barcha x nuqtalarida $f(x) < f(x_0)$ shartlar bajarilsa, x_0 nuqta $f(x)$ funksiyaning ... deyiladi.

a) lokal maksimumi b) lokal minimumi
c) o`zgarmasi d) nolga teng qiymati

7. Agar: 1) $f(x)$ funksiya x_0 nuqtada va uning qandaydir atrofida aniqlangan; 2) shu atrofning x_0 dan farqli barcha x nuqtalarida $f(x) > f(x_0)$ shartlar bajarilsa, x_0 nuqta $f(x)$ funksiyaning ... deyiladi.

a) lokal maksimumi b) lokal minimumi
c) o`zgarmasi d) nolga teng qiymati

8. Lokal maksimum va lokal minimum iboralari uchun umumiy nom ... iborasi ishlatiladi.

a) lokal ekstremum b) lokal o`suvchi
c) lokal kamayuvchi d) lokal o`zgarmas

9. Agar $f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo`lib, bu nuqta funsiyaning lokal ekstremumi bo`lsa, u holda ... bo`ladi.

a) $f'(x_0) = 0$ b) $f'(x) > 0$ c) $f'(x) < 0$ d) $f'(x) = 1$

10. Funksiyaning mavjud hosilasi nolga teng yoki hosilasi mavjud bo`lmagan nuqtalar shu funksiyaning ... deyiladi.

a) kritik nuqtalari b) maxsus nuqtalari
c) limit nuqtalari d) birlik nuqtalari

11. Agar son o`qida $x_0$ nuqtaning chap tomonidan o`ng tomoniga o`tishda $f'(x)$ hosila ishorasini plyusdan minusga o`zgartirsa, x_0 nuqta $f(x)$ ning ... nuqtasi bo`ladi.

a) lokal maksimum b) lokal minimum
c) o`zgarmas d) nolga teng qiymati

12. Agar son o`qida $x_0$ nuqtaning chap tomonidan o`ng tomoniga o`tishda $f'(x)$ hosila ishorasini minusdan plyusga o`zgartirsa, x_0 nuqta $f(x)$ ning ... nuqtasi bo`ladi.

a) lokal maksimum b) lokal minimum
c) o`zgarmas d) nolga teng qiymati

13. Agar son o`qida $x_0$ nuqtaning chap tomonidan o`ng tomoniga o`tishda $f'(x)$ hosila ishorasini o`zgartimasa, x_0 nuqta $f(x)$ ning ... nuqtasi bo`lmaydi.

a) lokal ekstremum b) limit
c) o`zgarmas d) nolga teng qiymatli

14. $f(x)$ funksiya x_0 nuqta atrofida ikki marta differensiallanuvchi va $f'(x_0) = 0$ bo`lsin. Agar $f''(x_0) > 0$ bo`lsa, x_0 nuqta $f(x)$ funksiyaning ... nuqtasi bo`ladi.

a) lokal maksimum b) lokal minimum
c) o`zgarmas d) nolga teng qiymati

15. $f(x)$ funksiya x_0 nuqta atrofida ikki marta differensiallanuvchi va $f'(x_0) = 0$ bo`lsin. Agar $f''(x_0) < 0$ bo`lsa, x_0 nuqta $f(x)$ funksiyaning ... nuqtasi bo`ladi.

a) lokal maksimum b) lokal minimum
c) o`zgarmas d) nolga teng qiymati

16. $f(x)$ funksiya $[a,\ b]$ kesmada aniqlangan va $x_0 \in [a,\ b]$ bo`lsin. Agarda barcha $x \in [a,\ b]$ uchun $f(x) \leq f(x_0)$ tengsizlik o`rinli bo`lsa, $f(x)$ funksiya x_0 nuqtada ... erishadi deyiladi.

a) eng katta qiymatga b) eng kichik qiymatga
c) lokal maksimum d) lokal minimum

17. $f(x)$ funksiya $[a,\ b]$ kesmada aniqlangan va $x_0 \in [a,\ b]$ bo`lsin. Agarda barcha $x \in [a,\ b]$ uchun $f(x) \geq f(x_0)$ tengsizlik o`rinli bo`lsa, $f(x)$ funksiya x_0 nuqtada ... erishadi deyiladi.

a) eng katta qiymatga b) eng kichik qiymatga
c) lokal maksimum d) lokal minimum

18. Agar $f(x)$ funksiya $[a,\ b]$ kesmada **uzluksiz** bo`lsa, u holda bu funksiya shu kesmada o`zining eng katta va eng kichik qiymatlariga erishadi.

a) eng katta va eng kichik qiymatlariga b) eng kichik qiymatlariga
c) eng katta qiymatlariga d) lokal minimum

19. Differensiallanuvchi $y = f(x)$ funksiyaning grafigi unga $(a,\ b)$ intervaldan olingan har bir nuqtada o`tkazilgan urinmadan pastda joylashgan bo`lsa, bu grafik shu intervalda ... deyiladi.

a) qavariq (yoki yuqoriga qavariq) b) botiq (yoki pastga qavariq)
c) lokal minimum d) lokal maksimum

20. Differensiallanuvchi $y = f(x)$ funksiyaning grafigi unga $(b,\ c)$ intervaldan olingan har bir nuqtada o`tkazilgan urinmadan tepada joylashgan bo`lsa, bu grafik shu intervalda ... deyiladi.

a) qavariq (yoki yuqoriga qavariq) b) botiq (yoki pastga qavariq)
c) lokal minimum d) lokal maksimum

21. Uzluksiz egri chiziqning qavariq qismini uning botiq qismidan ajratuvchi nuqta egri chiziqning ... deyiladi.

a) bukilish nuqtasi b) kritik nuqtasi
c) maksimum nuqtasi d) minimum nuqtasi

22. Agar $y = f(x)$ funksiya $(a,\ b)$ intervalning barcha nuqtalarida $f''(x) < 0$ bo`lsa, u holda bu intervalda $y = f(x)$ funksiya grafigi ... bo`ladi;

a) qavariq b) botiq c) o`suvchi d) kamayuvchi

23. Agar $y = f(x)$ funksiya $(a,\ b)$ intervalning barcha nuqtalarida $f''(x) > 0$ bo`lsa, u holda bu intervalda $y = f(x)$ funksiya grafigi ... bo`ladi.

a) qavariq b) botiq c) o`suvchi d) kamayuvchi

24. Agar $f''(x)$ ikkinchi hosila biror x_0 nuqtada nolga teng yoki mavjud bo`lmasa, hamda son o`qida x_0 nuqtaning bir tarafidan boshqa tarafiga o`tganda bu ikkinchi hosila ishorasini o`zgartirsa, u holda x_0 nuqta $y = f(x)$ funksiya grafigining ... bo`ladi.

a) bukilish nuqtasi b) kritik nuqtasi

c) maksimum nuqtasi d) minimum nuqtasi

25. $y = f(x)$ funksiya grafigi – $f(x)$ egri chiziq va l to`g`ri chiziq berilgan bo`lsin. $f(x)$ egri chiziqdagi $M$ nuqta koordinata boshidan egri chiziq bo`ylab cheksiz uzoqlashganda M nuqtadan l to`g`ri chiziqqacha masofa nolga intilsa, l to`g`ri chiziq bu $f(x)$ egri chiziqning ... deyiladi.

a) asimptotasi b) urinmasi c) normali d) kesuvchisi

26. Agar $\lim\limits_{x\to a} f(x) = \infty$, $\lim\limits_{x\to a-0} f(x) = \infty$, $\lim\limits_{x\to a+0} f(x) = \infty$ tengliklardan birontasi bajarilsa, u holda $x = a$ to`g`ri chiziq $y = f(x)$ egri chiziqning ... deyiladi.

a) vertikal asimptotasi b) gorizontal asimptota

c) urinmasi d) normali

27. Agar $\lim\limits_{x\to\infty} f(x) = k$ bo`lsa, u holda $y = f(x)$ egri chiziq $y = k$ ... ega deyiladi

a) vertikal asimptotasi b) gorizontal asimptota

c) urinmasi d) normali

28. $y = kx + b$ to`g`ri chiziq $y = f(x)$ egri chiziqning asimptotasi bo`lishi uchun $\lim\limits_{x\to\infty}(f(x) - kx - b) = 0$ tenglik bajarilishi zarur va yetarli.

a) $\lim\limits_{x\to\infty}(f(x) - kx - b) = 0$

b) $\lim\limits_{x\to\infty}(f(x) - kx - b) = 0$

c) $\lim\limits_{x\to\infty}(f(x) - kx - b) = 0$

d) $\lim\limits_{x\to\infty}(f(x) - kx - b) = 0$

29. $y = f(x)$ egri chiziqning $y = kx + b$ og`ma asimptotasida k koeffisiyent ... topiladi.

a) $k = \lim\limits_{x\to\infty}\frac{f(x)}{x}$,

b) $b = \lim\limits_{x\to\infty}(f(x) - kx)$

c) $k = \lim\limits_{x\to\infty}\frac{f(x)}{x} + 1$,

d) $b = \lim\limits_{x\to\infty}(f(x) - kx) - 1$

30. $y = f(x)$ egri chiziqning $y = kx + b$ og`ma asimptotasida b koeffisiyent ... topiladi.

a) $k = \lim\limits_{x\to\infty}\frac{f(x)}{x}$,

b) $b = \lim\limits_{x\to\infty}(f(x) - kx)$

c) $k = \lim\limits_{x\to\infty}\frac{f(x)}{x} + 1$,

d) $b = \lim\limits_{x\to\infty}(f(x) - kx) - 1$

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